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Aerodynamics. — *Théorie non-linéaire de l'anémomètre à fil chaud.* By R. BETCHOV. (Med. No. 61 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of February 26, 1949.)

Nous étudions ici les propriétés d'un anémomètre à fil chaud, en supposant que le transfert de chaleur du fil à l'air dépend d'une part de la différence de température et d'autre part du carré de cette différence. L'expérience confirme cette hypothèse et les conséquences peuvent être fort importantes. Cet effet de non-linéarité est plus grand que l'effet de conduction thermique.

I. *La loi de* KING *non-linéaire.*

La quantité de chaleur Q enlevée par seconde par un vent V à un fil de diamètre d et de longueur unité, est donnée par KING sous la forme:

$$Q = (a + b\sqrt{Vd})\,T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

où T désigne la différence de température entre le fil et l'air. Le calcul de KING, confirmé approximativement par l'expérience, donne:

$$a = \varkappa' \quad ; \quad b = \sqrt{2\pi\,\varkappa'\,\delta'\,c'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

avec $\varkappa' =$ conductivité thermique de l'air, $\delta' =$ densité de l'air, $c' =$ chaleur spécifique de l'air à volume constant.

Ces grandeurs peuvent varier avec T, et l'expérience montre que a augmente alors que b reste pratiquement constant. Intuitivement, on peut interpréter cet effet en disant que l'air en contact avec le fil est chauffé, ce qui augmente sa conductivité. En revanche, sa densité diminue, parce que la pression ne varie que très peu. Il faut admettre que les effets sur $\varkappa'$ et δ' se compensent, et que seul a varie.

Dans son mémoire original (Littérature, No. 1), KING signale que a augmente de 1,14 pour mille par degré; il décrit également un effet du diamètre sur ce terme a, effet dont nous ne discuterons pas ici. Il convient donc d'écrire:

$$Q = \{a(1 + \gamma\,T) + b\sqrt{Vd}\}\,T \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

où γ traduit la non-linéarité.

Il ne faut pas perdre de vue les hypothèses sur lesquelles repose le calcul de KING, qui admet que l'écoulement d'air se fait sans viscosité et que le flux de chaleur, tout près du fil, est constant. Il prend la chaleur spécifique

à volume constant, alors que la pression est certainement plus constante que la densité. C'est pourquoi nous considérons (3) comme une relation empirique, valable pour l'unité de longueur du fil, et nous souhaitons que le problème de KING fasse l'objet d'une étude plus approfondie.

Nous nous proposons ici d'étudier l'effet du terme γ sur les propriétés du fil chaud, et simplifions l'écriture en introduisant le nombre P tel que:

$$P = \frac{b}{a}\sqrt{Vd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$Q = a\,(1 + P + \gamma\, T)\, T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

II. *Equation générale du fil chaud.*

Nous prendrons les symboles suivants:

S = résistance du fil, par unité de longueur, à la température de fonctionnement,
S_0 = résistance du fil, par unité de longueur, à la température de l'air ambiant,
I = intensité du courant électrique de chauffage du fil,
a = coefficient de variation de S selon la température,
m = masse du fil, par unité de longueur,
c = chaleur spécifique du fil, en joule/gramme et degré,
$\varkappa$ = coefficient de conductivité thermique du fil en watt/cm et degré,
σ = section du fil,
l = demi-longueur du fil,
t = temps,
x = coordonnée de position, variant de $+l$ à $-l$.

Nous posons:

$$A = \frac{a}{aS_0}(1 + P) = \frac{a + b\sqrt{Vd}}{aS_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

L'équation du fil chaud doit exprimer l'équilibre entre la chaleur fournie par seconde, la chaleur de KING emportée par le vent, la chaleur nécessaire à élever la température du fil et la chaleur transmise par conduction. On obtient:

$$SI^2 = A(S - S_0) + \frac{a\gamma}{a^2 S_0^2}(S - S_0)^2 + \frac{mc}{aS_0}\frac{\partial S}{\partial t} - \frac{\varkappa\sigma}{aS_0}\frac{\partial^2 S}{\partial x^2} \quad . \quad . \quad . \quad (7)$$

En régime continu, et en introduisant les grandeurs:

$$\left.\begin{aligned} y = x/l^* \;\; ; \;\; l^* = \sqrt{\frac{\varkappa\sigma}{aS_0(A - I^2)}} \;\; ; \;\; Z = \frac{A - I^2}{S_0 I^2}(S - S_0) \\ G = \frac{2}{3}\frac{a\gamma}{a^2 S_0}\frac{I^2/A^2}{(1 - I^2/A)^2} = \frac{2}{3}\frac{\gamma}{a}\frac{1}{1 + P}\frac{I^2/A}{(1 - I^2/A)^2} \end{aligned}\right\} \quad (8)$$

on obtient l'équation (7) sous la forme:

$$Z + 3/2\, G Z^2 - \frac{\partial^2 Z}{\partial y^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

III. *Intégration exacte du cas statique.*

En multipliant (9) par $\partial z/\partial y$ et intégrant, on obtient, avec une constante:

$$\partial z/\partial y = \sqrt{G Z^3 + Z^2 - 2 Z + cste.} \quad . \quad . \quad . \quad . \quad (10)$$

On voit que $\partial z/\partial y$ est nulle pour une valeur négative de Z et peut l'être pour deux valeurs positives de Z. Aux extrêmités du fil, on a $S = S_0$ et $Z = 0$; au centre il faut que Z soit positif et $\partial z/\partial y$ nul. Le domaine qui nous intéresse est donc compris entre $Z = 0$ et la première racine positive donnant $\partial z/\partial y = 0$, que nous désignerons par $Z = B$. Posons:

$$Z(y) = B - X^2(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

En vertu de la relation:

$$G B^3 + B^2 - 2 B + cste = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

on obtient:

$$4 (\partial x/\partial y)^2 = - G X^4 + E X^2 + D \quad . \quad . \quad . \quad . \quad (13)$$

avec:

$$E = 1 + 3 GB \quad ; \quad D = 2 (1 - B) - 3 G B^2. \quad . \quad . \quad . \quad (14)$$

Nous considérerons désormais B comme une nouvelle constante d'intégration, donnant la température au milieu du fil. Au centre du fil, on a $X = 0$ et:

$$4 (\partial x/\partial y)^2 = D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

ce qui montre que D est positif, et en général petit. Aux extrêmités du fil $Z = 0$ et $X = \pm \sqrt{B}$.

Les racines de l'équation (13) sont:

$$X^2 = \frac{- E \pm \sqrt{E^2 + 4 GD}}{- 2 G}. \quad . \quad . \quad . \quad . \quad (16)$$

En introduisant le paramètre β, tel que:

$$\text{sin hyp}\, \beta = sh\, \beta = \frac{2 \sqrt{GD}}{E} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

on peut mettre (13) sous la forme:

$$4 (\partial x/\partial y)^2 = G \left[\frac{E}{2G} (ch\, \beta + 1) - X^2 \right] \left[\frac{E}{2G} (ch\, \beta - 1) + X^2 \right]. \quad . \quad (18)$$

Définissons l'angle φ, fonction de y, tel que:

$$X = \sqrt{\frac{D}{E\, ch\, \beta}} \frac{\sin \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad . \quad . \quad . \quad . \quad (19)$$

avec:

$$k^2 = \frac{ch\beta + 1}{2\,ch\beta} \quad \ldots\ldots\ldots \quad (20)$$

L'équation (18) devient alors:

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \tfrac{1}{2}\sqrt{E\,ch\beta}\;dy \quad \ldots\ldots \quad (21)$$

et nous obtenons l'intégrale elliptique de première espèce:

$$U(k;\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \tfrac{1}{2}\sqrt{E\,ch\beta}\;y \quad \ldots\ldots \quad (22)$$

La variable y varie de $-\xi$ à $+\xi$, avec:

$$\xi = l/l^* \quad \ldots\ldots\ldots \quad (23)$$

et nous avons, pour $y = \xi$ et $X^2 = B$:

$$\frac{\xi}{2}\sqrt{E\,ch\beta} = \int_0^{\varphi_{max}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad \ldots\ldots \quad (24)$$

avec:

$$\sqrt{B} = \sqrt{\frac{D}{E\,ch\beta}}\,\frac{\sin\varphi_{max}}{\sqrt{1-k^2\sin^2\varphi_{max}}} \quad \ldots\ldots \quad (25)$$

Cette dernière équation peut s'écrire:

$$\frac{1}{\sin^2\varphi_{max}} = k^2 + \frac{D}{EB\,ch\,\beta} \quad \ldots\ldots \quad (26)$$

A partir de (20), on peut déduire:

$$\frac{1}{k^2} = 1 + Tgh^2(\beta/2) \quad \ldots\ldots \quad (27)$$

Lorsque β va de 0 à $+\infty$, k^2 varie de 1 à 0,5.

Si l'on connait donc G et B, on peut calculer successivement D, E, β, k et φ_{max}. Une table de $U(k,\varphi)$ nous permet alors de calculer ξ. La figure 1 donne les résultats obtenus par ce procédé et permet, en partant d'un fil donné et en connaissant γ, de déterminer B à partir de G et de ξ.

La répartition de la température sur le fil est donnée par Z en fonction de y, soit par X fonction de φ et φ fonction de y. La relation $\varphi(y)$ est donnée par le quotient de (22) et (24), soit:

$$\frac{y}{\xi} = \frac{\displaystyle\int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}}{\displaystyle\int_0^{\varphi_{max}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}} \quad \ldots\ldots \quad (28)$$

A partir de (19) et (25), on obtient une relation entre φ et X, soit:

$$\frac{X^2}{B} = \frac{\sin^2\varphi}{\sin^2\varphi_{max}} \frac{1-k^2\sin^2\varphi_{max}}{1-k^2\sin^2\varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

La résistance totale R du fil est donnée par:

$$R = \int_{-l}^{+l} S\,dx = \frac{2S_0 I^2}{A-I^2} l^* \int_0^{\xi} Z\,dy + 2S_0 l \quad . \quad . \quad . \quad . \quad . \quad (30)$$

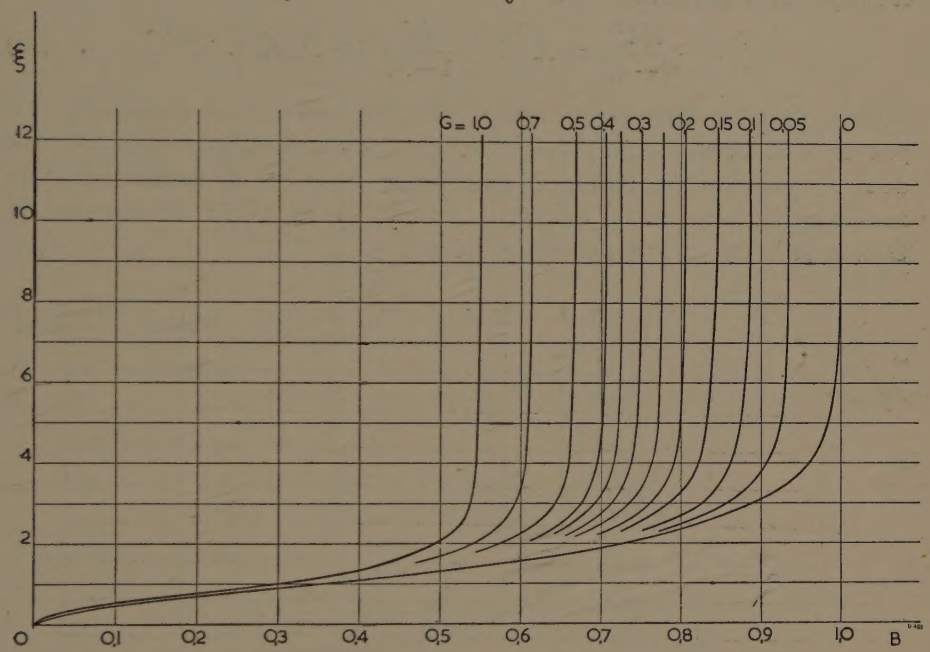

Fig. 1. Relation entre les grandeurs B, G et ξ.

Introduisons la résistance à froid R_0 et la fonction X:

$$R - R_0 = \frac{R_0 I^2}{A-I^2}\left(B - \frac{1}{\xi}\int_0^{\xi} X^2\,dy\right) \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Remplaçons X selon (19) et dy selon (21), soit:

$$R - R_0 = \frac{R_0 I^2}{A-I^2}\left(B - \frac{2D}{\xi (E\,ch\,\beta)^{3/2}}\int_0^{\varphi_{max}} \frac{\sin^2\varphi}{(1-k^2\sin^2\varphi)^{3/2}}\,d\varphi\right) \quad . \quad . \quad (32)$$

L'intégrale est égale à $2\,\partial U/\partial k^2$ et nous obtenons:

$$R - R_0 = \frac{R_0 I^2}{A-I^2}\left(B - \frac{2D}{E\,ch\,\beta}\frac{\partial U/\partial k^2}{U}\right) \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Les valeurs de $\partial U/\partial k^2$ peuvent être déduites d'une bonne table de $U(k, \varphi)$ avec une approximation suffisante.

Si le fil était parfait, l'expression entre ( ) dans (33) serait à remplacer par l'unité; nous introduirons donc la grandeur M telle que:

$$M = 1 - B + \frac{2D}{E\,ch\,\beta}\,\frac{\partial U/\partial k^2}{U} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

La formule (33) nous donne alors:

$$R = R_0\,\frac{1 - M\,I^2/A}{1 - I^2/A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

et l'importante relation:

$$\frac{R\,1^2}{R - R_0} = A\left\{1 + \frac{M}{1-M}\,(1 - I^2/A)\right\} \quad . \quad . \quad . \quad . \quad (36)$$

Cette dernière équation permet un contrôle facile, parce que R, R_0 et I peuvent être mesurés avec précision, et que les courbes obtenues en fonction de R/R_0 par exemple, indiquent directement les effets de conduction et de non-linéarité. Nous avons calculé les valeurs de $M/1-M$, en fonction de G et ξ, et la figure 2 donne nos résultats.

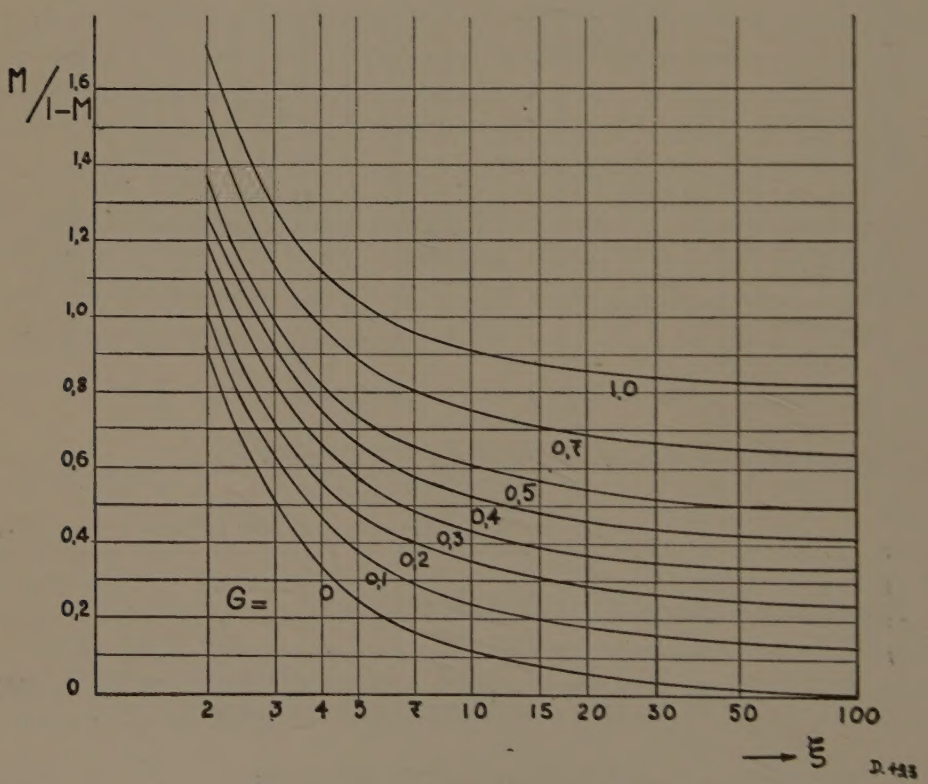

Fig. 2. Valeurs de $M/1-M$, selon G et ξ.

Une bonne approximation est donnée par:

$$\frac{M}{1-M} = \frac{1}{B_0} - 1 + \frac{1}{\xi - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

où B_0 correspond à B, dans le cas $\xi = \infty$, soit:

$$B_0 = \frac{+\sqrt{1+6G}-1}{3G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

IV. *Quelques approximations utiles.*

Nous avons remarqué, en effectuant les calculs nécessaires au tracé de la fig. 1, que l'on peut donner à k la valeur 1 sans introduire de grandes erreurs.

Cela implique $\beta = 0$ et (19) donne alors:

$$X = \sqrt{\frac{D}{E}}\, Tg\, \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

L'intégrale (22) devient:

$$U = \int_0^{\varphi} \frac{d\varphi}{\cos\varphi} = \tfrac{1}{2} L \frac{1+\sin\varphi}{1-\sin\varphi} = \tfrac{1}{2}\sqrt{E}\, y \quad . \quad . \quad . \quad . \quad (40)$$

d'où l'on déduit:

$$ch\,(2U) = \frac{1+\sin^2\varphi}{1-\sin^2\varphi} = ch\,(\sqrt{E}\, y) \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$X^2 = \frac{D}{2E}\,(ch\,\sqrt{E}\, y - 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Aux limites, on a:

$$B = \frac{D}{2E}\,(ch\,\sqrt{E}\,\xi - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

ce qui donne:

$$Z = \frac{B}{1 - 1/ch\,\sqrt{E}\,\xi}\left(1 - \frac{ch\,\sqrt{E}\, y}{ch\,\sqrt{E}\,\xi}\right). \quad . \quad . \quad . \quad . \quad (44)$$

Pour calculer M, il faut prendre:

$$\lim_{k \to 1} \partial U / \partial k^2 = \frac{1}{2}\int_0^{\varphi} \frac{\sin^2\varphi}{\cos^3\varphi}\, d\varphi = \frac{1}{4}\left(\frac{\sin\varphi}{\cos^2\varphi} - U(\varphi)\right) \quad . \quad . \quad (45)$$

ce qui donne:

$$M = 1 - \frac{B}{1 - 1/ch\,\sqrt{E}\,\xi}\left(1 - \frac{Th\,\sqrt{E}\,\xi}{\sqrt{E}\,\xi}\right). \quad . \quad . \quad . \quad . \quad (46)$$

On voit que la non-linéarité remplace ξ par $\sqrt{E}\,\xi$, et abaisse la température centrale.

Si l'on prend (44) comme solution de (9), on voit que l'équation est

satisfaite, à un terme en $(ch\sqrt{E}\,y/ch\sqrt{E}\,\xi)^2$ près, et que B est donné à peu près par:

$$B=\left(\frac{\sqrt{1+6G}-1}{3G}\right)(1-1/ch\sqrt{E}\,\xi) \quad . \quad . \quad . \quad . \quad (47)$$

avec

$$E\simeq\sqrt{1+6G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Prenons un exemple qui correspond à un cas extrême. Soit un fil de Pt avec 10 % d'Ir, diamètre de 7 microns, longueur $2l = 1{,}14$ mm. Exposé à un vent de 5 m/sec et chauffé avec 75 mA, il nous donne:

$$P=2{,}9 \quad ; \quad A=1{,}2 \quad 10^{-2} \quad ; \quad I^2/A=0{,}47$$
$$l^*=0{,}15\text{ mm} \quad ; \quad \xi=3{,}75.$$

En prenant $\gamma = 1{,}2.\ 10^{-3}$, et à l'aide des fig. 1 et 2 nous avons déterminé:

$$G=0{,}25 \quad ; \quad B=0{,}76 \quad ; \quad E=1{,}56 \quad ; \quad M=0{,}4.$$

Les autres paramètres ont les valeurs calculées suivantes:

$$k=0{,}99756 \quad ; \quad \varphi_{max}=79{,}5 \text{ degrés};$$
$$sh\,\beta=0{,}14 \quad ; \quad D=0{,}0477 \quad ; \quad U=2{,}34.$$

En fig. 3 nous donnons le profil des températures, calculé exactement, le profil calculé avec l'approximation $k = 1$, et le profil calculé avec $\gamma = 0$.

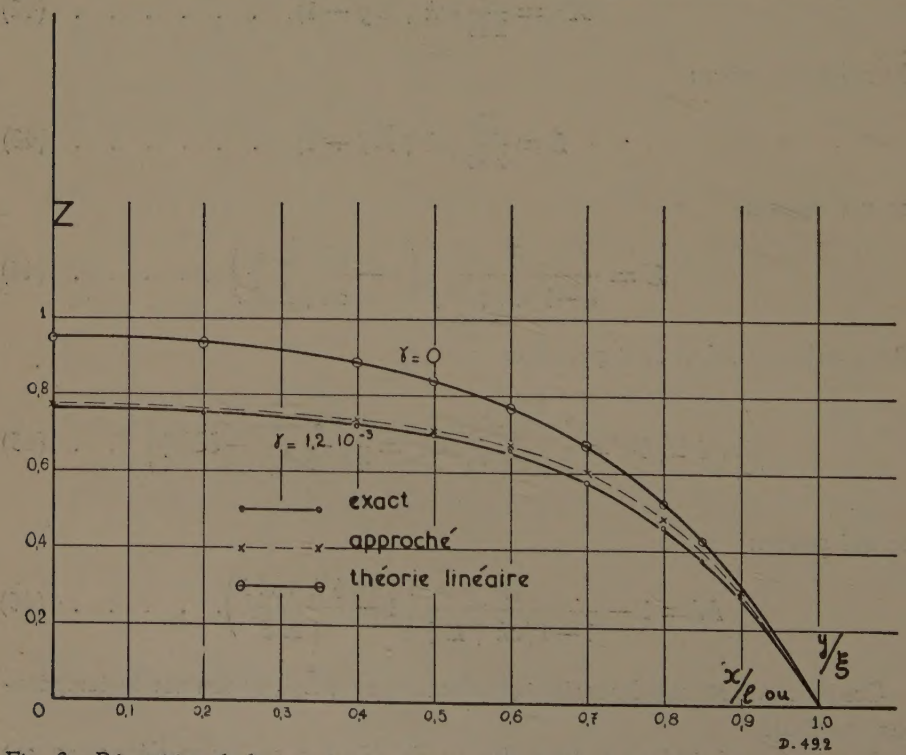

Fig. 3. Répartition de la température, avec un fil de 7 microns, selon la théorie linéaire et selon nos formules.

On peut remarquer que la non-linéarité donne une distribution plus régulière de la température, et que l'approximation est suffisante.

A partir de (35), on calcule $R/R_0 = 1{,}5$, alors que le calcul avec $\gamma = 0$ donnerait 1,9. La température moyenne donnant $R/R_0 = 1{,}5$ serait 380 degrés.

Quant au terme $RI^2/R—R_0$, il passe de la valeur 1,24 A, lorsque le courant est très faible, à la valeur 1,35 A lorsque I atteint 75 mA. Il varie donc de environ 10 % entre $R/R_0 = 1$ et $R/R_0 = 1{,}5$, ce qui est de l'ordre des variations observées.

Ainsi, cette grandeur est constante à 10 %, et la loi de KING est vérifiée, mais nous verrons plus loin que l'inertie thermique est très différente de la valeur attendue.

V. *L'équation dynamique du fil chaud.*

Pour calculer la variation de la résistance totale du fil, lorsque le courant ou le vent fluctuent, il faut remonter à l'équation (7). En y remplaçant I, S et V (contenu dans le terme A), par $I + i e^{j\omega t}$, $S + s e^{j\omega t}$ et $V + v e^{j\omega t}$ on obtient, après suppression des termes d'ordre zéro, deux et plus, ainsi que du facteur $e^{j\omega t}$:

$$\left.\begin{aligned} 2SIi + I^2 s = \frac{aP}{2\alpha S_0}\frac{v}{V}(S-S_0) + As + \frac{2a\gamma}{\alpha^2 S_0^2}(S-S_0)s + \\ + \frac{mc}{\sigma S_0} j\omega s - \frac{\varkappa\sigma}{\alpha S_0}\frac{\partial^2 s}{\partial x^2}. \end{aligned}\right\} \quad . \quad . \quad (49)$$

Introduisons:

$$z = s\frac{A-I^2}{S_0 I^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

et, identique à la formule (9) de notre Med. 55:

$$\omega^* = \frac{\alpha S_0}{mc}(A-I^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

On obtient alors, après introduction de Z:

$$2\frac{i}{I} + 2\frac{i}{I}\frac{I^2/A}{1-I^2/A}Z - \frac{1}{2}\frac{P}{1+P}\frac{Z}{1-I^2/A}\frac{v}{V} = z(1+3GZ+j\omega/\omega^*) - \frac{\partial^2 z}{\partial y^2}. \quad (52)$$

La fonction $Z(y)$ figure deux fois dans cette équation différentielle, et il nous faut utiliser l'approximation $k = 1$ pour éviter de grandes complications. En prenant l'expression de Z selon (44), on obtient:

$$-\frac{\partial^2 z}{\partial y^2} + z\left(1 + \frac{3GB}{1-1/ch\sqrt{E}\xi} + j\omega/\omega^* - \frac{3GB}{ch\sqrt{E}\xi - 1}ch\sqrt{E}y\right) = C_1 + C_2\frac{ch\sqrt{E}y}{ch\sqrt{E}\xi} \quad (53)$$

avec:

$$C_1 = 2\frac{i}{I}\left(1+\frac{I^2/A}{1-I^2/A}\frac{B}{1-1/ch\sqrt{E}\,\xi}\right)-\frac{1}{2}\frac{v}{V}\left(\frac{P}{1+P}\frac{1}{1-1^2/A}\frac{B}{1-1/ch\sqrt{E}\,\xi}\right) \quad (54)$$

$$C_2 = -2\frac{i}{I}\frac{1^2/A}{1-1^2/A}\frac{B}{1-1/ch\sqrt{E}\,\xi}+\frac{1}{2}\frac{v}{V}\left(\frac{P}{1+P}\frac{1}{1-1^2/A}\frac{B}{1-1/ch\sqrt{E}\,\xi}\right). \quad (55)$$

La solution sans second membre de (53) est une fonction de MATHIEU, prise pour une valeur purement imaginaire de l'argument. Mais si $\sqrt{E}\,\xi$ est assez grand, supérieur à 4 par exemple, on peut négliger le terme en $ch\ \sqrt{E}\,y$ du membre de gauche de (53). En effet, il n'est important que lorsque y est voisin de ξ; mais nous prendrons comme condition aux limites $z(\xi) = 0$, et le produit $z\ ch\ \sqrt{E}\,y$ ne sera jamais important. Nous négligerons également le terme $1/ch\ \sqrt{E}\,\xi$ dans le dénominateur d'un des termes de gauche de (53), ce qui revient à prendre $Z = B$ dans le terme facteur de z.

Compte tenu de la définition de E, on a alors:

$$-\frac{\partial^2 z}{\partial y^2}+z\,(E+j\,\omega/\omega^*) = C_1 + C_2\,\frac{ch\sqrt{E}\,y}{ch\sqrt{E}\,\xi} \quad . \quad . \quad . \quad (56)$$

dont la solution, nulle pour $y = \pm\xi$, est:

$$z = \frac{C_1}{Ep^2}\left(1-\frac{ch\,p\sqrt{E}\,y}{ch\,p\sqrt{E}\,\xi}\right)+\frac{C_2}{j\,\omega/\omega^*}\left(\frac{ch\sqrt{E}\,y}{ch\sqrt{E}\,\xi}-\frac{ch\,p\sqrt{E}\,y}{ch\,p\sqrt{E}\,\xi}\right). \quad . \quad (57)$$

avec

$$p = \sqrt{1+j\,\omega/E\,\omega^*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

On remarque que la non-linéarité remplace la fréquence propre ω^* par une nouvelle fréquence plus élevée d'un facteur E (Comparer avec (8) de notre article donné sous Littérature, No 5). Les amplitudes selon C_1 et C_2 sont aussi modifiées par la non-linéarité.

La variation r de la résistance totale R sera:

$$r = \int_{-l}^{+l} s\,dx = \frac{R_0 I^2}{A-I^2}\,\frac{1}{\xi}\int_0^{\xi} z\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

L'intégration donne:

$$r = \frac{R_0 I^2}{A-I^2}\left\{\frac{C_1}{Ep^2}\left(1-\frac{Tgh\,p\sqrt{E}\,\xi}{p\sqrt{E}\,\xi}\right)+\frac{C_2}{j\,\omega/\omega^*}\,\frac{1}{\sqrt{E}\,\xi}\left(Tgh\sqrt{E}\,\xi-\frac{Tgh\,p\sqrt{E}\,\xi}{p}\right)\right\}. \quad (60)$$

Nous avons supposé plus haut que $\sqrt{E}\,\xi$ était assez grand, aussi pouvons nous donner aux tangentes hyperboliques la valeur 1 (la présence d'un argument complexe est ici sans importance).

On obtient alors:

$$r = \frac{R_0 I^2}{A - I^2} \left\{ \frac{C_1}{E p^2} (1 - 1/p\sqrt{E}\,\xi) + \frac{C_2}{j\,\omega/\omega^*} \frac{1}{\sqrt{E}\,\xi} \left(\frac{p-1}{p} \right) \right\} . \quad . \quad . \quad (61)$$

VI. *La réponse à une fluctuation du courant.*

Prenons $v = 0$ dans (54) et (55), on peut alors transformer (61) en supprimant les termes en $1/ch\sqrt{E}\,\xi$:

$$r\,I = \frac{2\,i\,R_0\,I^2/A}{(1 - I^2/A)^2} \left\{ \frac{(1 - I^2/A\,(1-B))(1 - 1/p\sqrt{E}\,\xi}{E\,p^2} - \frac{B\,I^2/A}{\sqrt{E}\,\xi} \frac{\omega^*}{j\,\omega} \left(\frac{p-1}{p} \right) \right\} . \quad (62)$$

Cette formule donne la tension alternative produite aux bornes du fil chaud par le courant de modulation i, en plus de la tension normale $R\,i$. Bien que cette formule paraisse compliquée, elle peut être adaptée aux besoins pratiques. Si la fréquence $\omega/2\pi$ tend vers l'infini, (62) devient:

$$r\,I = \frac{2\,i\,R_0\,I^2/A}{(1 - I^2/A)^2} \frac{(1 - I^2/A\,(1-B) - I^2/A\,B/\sqrt{E}\,\xi)}{j\,\omega/\omega^*} . \quad . \quad . \quad (63)$$

Compte tenu de (35) donnant R, (46) donnant M et (51) donnant ω^*, on trouve:

$$r\,I = \frac{-j}{\omega/2\pi}\,i\,C\,R\,I^2 \qquad \text{avec} \qquad C = \frac{a\,S_0}{\pi\,m\,c} . \quad . \quad . \quad (64) \quad (65)$$

La constante C correspond à celle de nos précédentes publications, et (64) montre que la tension $r\,I$ prise en haute fréquence permet de mesurer C, sans être gêné par la conduction ni la non-linéarité. La méthode décrite sous Litt. 4 est donc plus indiquée que celle qui consiste à mesurer des déphasages, avec ω de l'ordre de ω^*.

On peut directement vérifier ce point en prenant ω très grand dans l'équation (49), ce qui rend négligeable l'effet des termes de conduction et de non-linéarité.

Lorsque ω tend vers zéro, la tension devient:

$$r\,I = \frac{2\,i\,R_0\,I^2/A}{(1 - I^2/A)^2} \frac{1}{E} \left\{ 1 - \frac{I^2}{A} M - \frac{1}{\sqrt{E}\,\xi} (1 - I^2/A\,(1 - \tfrac{1}{2}B)) \right\} . \quad (66)$$

La fonction complexe $r\,I$ selon (62) doit donc passer de (64) à (66) lorsque ω varie de 0 à une grande valeur. Le tracé complexe de cette fonction donne pratiquement un demi-cercle, ce qui permet de poser approximativement:

$$r\,I = \frac{r\,I\,(\omega = 0)}{1 + j\,\omega/\omega^{**}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

où ω^{**} désigne la fréquence propre effective. En fig. 4 nous avons tracé le demi-cercle et indiqué quelques valeurs de ω/ω^{**}. A partir de (62), et dans le cas de l'exemple ci-dessus, nous avons calculé les tensions $r\,I$ pour

quelques valeurs de $\omega/E\,\omega^*$. On voit que les deux fonctions se confondent, en basses fréquences, si l'on prend $\omega^{**} = 1{,}1\,E\,\omega^* = 1{,}72\,\omega^*$. En haute fréquence, (64) et (67) doivent être égales, ce qui permet de calculer une

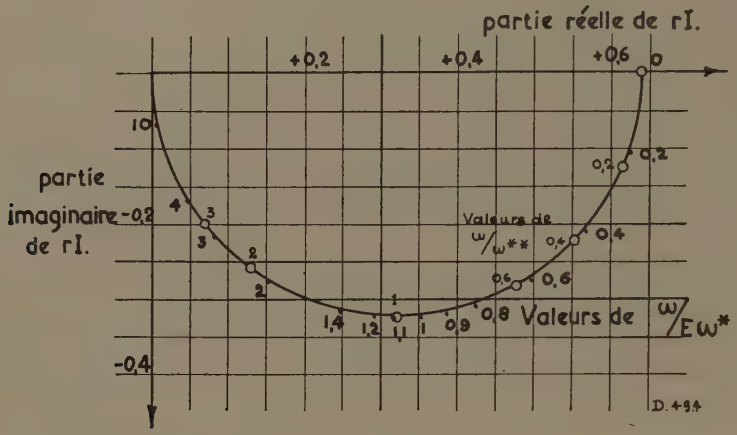

Fig. 4. Etude de la tension $r\,I$.
Le demi-cercle correspond à la formule (67); les points donnent $r\,I$, selon (62), pour quelques valeurs de $\omega/E\omega^*$.

valeur de ω^{**} satisfaisante, lorsque l'effet d'inertie thermique est important. On obtient:

$$\omega^{**} = \frac{E\,\omega^*}{1 - 1/\sqrt{E}\,\xi \left\{ \dfrac{1 - I^2/A\,(1 - {}^1/_2\,B)}{1 - I^2/A\,M} \right\}} \quad \ldots\ldots \quad (68)$$

Dans le cas traité on trouve $\omega^{**} = 1{,}23\,E\,\omega^*$, soit $\omega^{**} = 1{,}92\,\omega^*$: la fréquence propre effective est presque le double de la valeur attendue. L'approximation (67) donne donc un tracé correcte de la fonction, mais les phases selon (68) ne seront exactes qu'à 10 %.

Le dénominateur de (68) dépend principalement de ξ, et il élève la fréquence propre. Au lieu de se compenser, comme dans le cas statique, les deux effets réduisent l'inertie thermique.

Intuitivement, on peut dire que la conduction raccourcit la partie chaude du fil, et réduit ainsi la chaleur nécessaire pour modifier la température centrale; la non-linéarité tient à la présence d'air chaud autour du fil, et l'inertie thermique de l'air est négligeable, ce qui améliore la réponse globale de l'anémomètre.

Lorsque le fil est poussiéreux, la quantité d'air immobile est plus forte, et l'expérience montre que le terme a en (3) est élevé, alors que b reste inchangé. Il faut donc prévoir une action dynamique de la poussière du fil, dans la mesure où E est modifié. L'effet dynamique peut être plus important que l'effet statique.

Le fil de l'exemple cité nous montre que, avec un terme $R\,I^2/R - R_0$

constant à 10 %, la fréquence propre peut être presque double de la valeur normalement prévue.

VII. *La réponse à une fluctuation du vent.*

En prenant $i = 0$ dans les formules (54) et (55), et en maintenant v, en peut transformer (61) on:

$$rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_0 I^3/A}{(1-I^2/A)^2}\frac{B}{1-1/ch\sqrt{E}\xi}\left\{\frac{1-1/p\sqrt{E}\xi}{Ep^2} - \frac{\omega^*}{j\omega}\frac{p-1}{p}\frac{1}{\sqrt{E}\xi}\right\} \quad . \quad . \quad . \quad . \quad (69)$$

Avec ω tendant vers zéro, on a:

$$rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_0 I^3/A}{(1-I^2/A)^2}\frac{B}{1-1/ch\sqrt{E}\xi}\left\{\frac{1-\frac{3}{2}\frac{1}{\sqrt{E}\xi}}{E}\right\} \quad . \quad . \quad (70)$$

Si ω tend vers l'infini, on a:

$$rI = -\frac{1}{2}\frac{v}{V}\frac{P}{1+P}\frac{R_0 I^3/A}{(1-I^2/A)^2}\frac{B}{1-^1/ch\sqrt{E}\xi}\left\{\frac{1-\frac{1}{\sqrt{E}\xi}}{j\omega/\omega^*}\right\} \quad . \quad . \quad (71)$$

et on peut remplacer approximativement (69) par le demi-cercle:

$$rI = \frac{rI(\omega = 0)}{1 + j\omega/\omega^{**}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

avec:

$$\omega^{**} = E\,\omega^* \frac{1 - 1/\sqrt{E}\xi}{1 - {}^3/_2 1/\sqrt{E}\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

ce qui dans le cas de l'exemple traité donne $\omega^{**} = 1{,}81\ \omega^*$.

Il n'y a donc pas, en principe, égalité entre la réponse dynamique à une variation i et à une variation v.

Cette difficulté est introduite par le terme $\sqrt{E}\xi$, soit par la conduction; et la non-linéarité tend à en diminuer l'importance (facteur $\sqrt{E}$).

Nous espérons publier prochainement quelques résultats d'expériences, et le calcul des écarts signalés par divers auteurs.

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Zoology. — *Zonophryxus dodecapus nov. spec., a remarkable species of the family Dajidae (Crustacea Isopoda) from the Canary Islands.* By L. B. HOLTHUIS. (Communicated by Prof. C. J. VAN DER KLAAUW.)

(Communicated at the meeting of February 26, 1949.)

(31st Contribution to the Knowledge of the Fauna of the Canary Islands edited by Dr. D. L. UYTTENBOOGAART, continued by Dr. C. O. VAN REGTEREN ALTENA.) [1])

During a trip to the Canary Islands in the spring of 1947, several Caridean Crustacea were collected by Dr. G. THORSON of Universitetets Zoologiske Museum, Copenhagen. Dr. THORSON was so kind as to entrust me with the study of this material, a report on which is now in the press.

During the examination of the Caridea, a number of Isopod parasites of *Parapandalus narval* (Fabr.) were found in a jar containing several specimens of this pandalid prawn. One of the parasites was still attached to the body of its host. Examination of these Isopods showed them to belong to a new species of the genus *Zonophryxus,* which proved to be highly interesting in many respects. The present paper deals with this new species.

Zonophryxus dodecapus nov. spec.

Material examined: Los Cristianos, Tenerife, Canary Islands; on specimens of *Parapandalus narval* (Fabr.); rocky bottom, depth 210 m; March 26, 1947; G. THORSON leg. — 5 ♀♀ and 2 ♂♂.

Situation: The female is situated on the middorsal region of the posterior half of the carapace of the Decapod prawn *Parapandalus narval* (Fabr.) (= *Parapandalus pristis* (Risso)). The longitudinal axis of the parasite is parallel to that of its host, but the head of the parasite is directed to the tail end of the host.

Size: The females are 9 to 12 mm long, 5 to 7mm broad, and 2 to 3 mm high. The males are about 3 mm long.

♀. The body of the female is oval in outline, with the posterior end truncate. The dorsal surface (fig. 1*b*) is very convex, especially in the posterior and lateral parts, where the surface almost vertically (or even with a distinctly convex curve) dips down to the ventro-lateral border; anteriorly it slopes more gradually to this border. On the dorsal surface the various thoracic segments are rather distinctly defined. The second and third segments are least distinctly indicated, but the fourth to sixth are well marked. The first segment is completely fused with the cephalon, while the separation of the seventh thoracic segment from the abdomen is obscure,

[1]) Contribution 26 to 30 are published in Tijdschrift voor Entomologie, vol. 91.

if visible at all. The ventro-lateral border of the body (fig. 1*a*), which is one of the most characteristic features of the present genus, is oval, with the posterior end truncate. Small notches in this border are visible on the lines separating the various thoracic segments. The anterior half of the

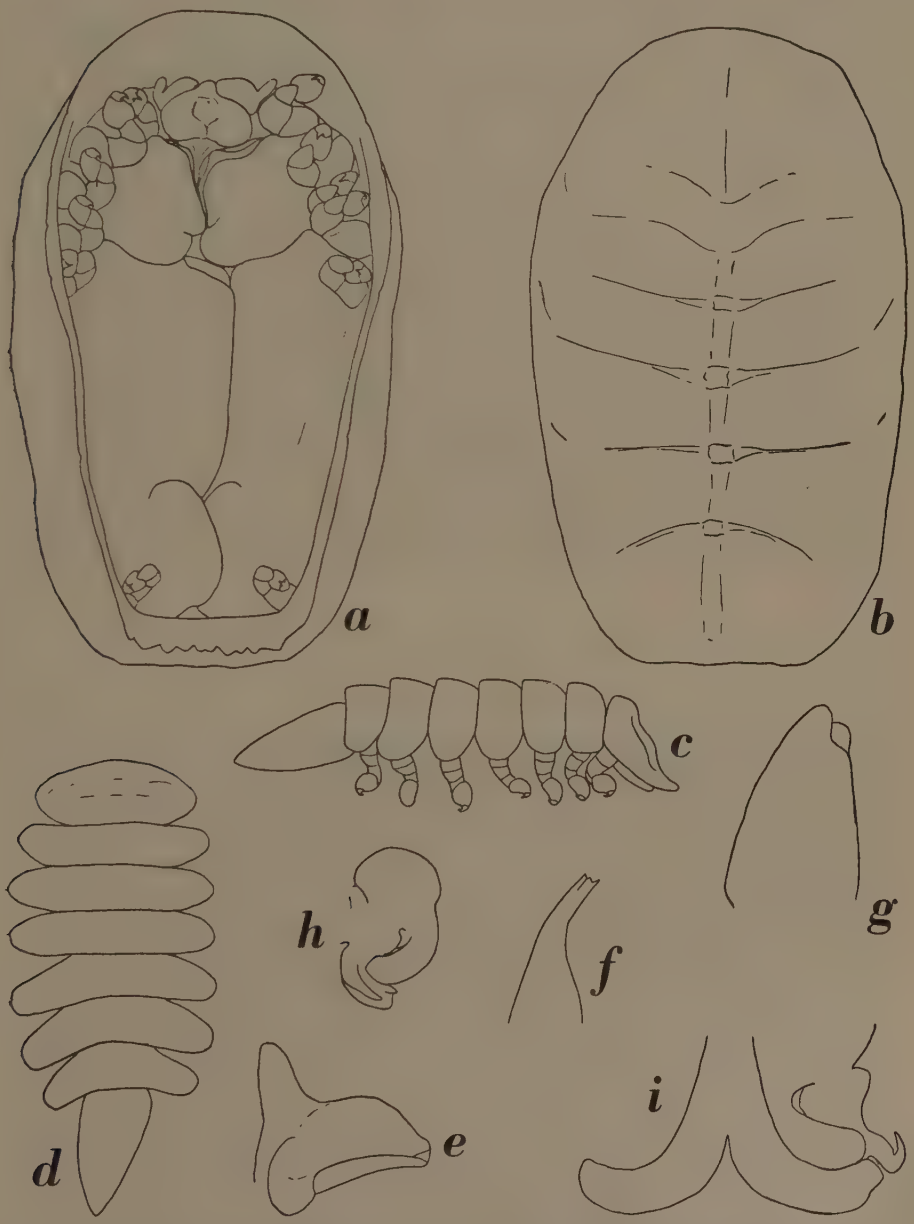

Fig. 1. *Zonophryxus dodecapus* nov. spec. *a*, body of female in ventral view; *b*, body of female in dorsal view; *c*, body of male in lateral view; *d*, body of male in dorsal view; *e*, internal and external antennae; *f*, mandible; *g*, left plate of lower lip; *h*, maxilliped; *i*, sternum in ventral view, with part of left maxilliped showing. *a*, *b*, × 7; *c*—*f*, *i*, × 25; *g*, × 50; *h*, × 10. (*e*—*i* of a female specimen.)

border shows four such notches at each side. The anterior of these notches is situated on the line separating the cephalon (with which the first thoracic segment is fused) from the second thoracic segment; the other three notches are placed on the lines which separate the second segment from the third, the third from the fourth, and the fourth from the fifth segment. In the posterior half of each lateral side of the border only one notch is visible: this notch is situated in about the middle of the posterior half of the border, on the line separating the fifth from the sixth thoracic segments. The truncated posterior part of the border bears 10 rather irregular teeth, which in one of the specimens are somewhat inconspicuous. As already remarked by RICHARDSON (1906) these teeth probably define the various abdominal segments. The ventral side of the body (fig. 1*a*) is strongly concave. This concavity is almost entirely covered by the oostegites of the pereiopods, so that it is not distinct at first, but may be seen when the oostegites are removed. The external antenna (fig. 1*e*) consists of a single papilliform joint. The internal antennae are formed by two broad lobes, which enclose the oral cone. Of the mouth parts the mandible (fig. 1*f*) is slender and ends in three small teeth. Behind and pressed against the oral cone are two oval plates, which touch each other in the middle and have a lobe at the inner side near the top. These two plates (one of which is figured here, fig. 1*g*) are considered by BARNARD to form the lower lip. Between the thoracic legs the ventral surface of the body bears the sternal process (fig. 1*i*), which is covered by the oostegites of the first pair of pereiopods. This process is anchor-shaped, the two flukes are curved and directed laterally, while the top is directed anteriorly; the posterior margin of this anchor is deeply incised in the middle. At each side of the sternal process and pressed close to the inner curve of the flukes lie the maxillipeds (fig. 1*h*). The maxillipeds consist of a thin anterior plate, which posteriorly ends in two broad curved processes, the inner of which curves outwards, the outer curves inwards, the processes partly covering each other. The outer process bears a sort of palp. BARNARD (1913) mentions the presence in one of his specimens of *Zonophryxus quinquedens* of a maxilla, which should be situated below the maxilliped. No such maxilla could be found by me in the specimens examined. There are six pairs of pereiopods. In this feature the present species differs from all Dajidae, except for the species *Paradajus tenuis* Nierstrasz & Brender à Brandis. The first five pereiopods are placed close together in the anterior half of the body, the sixth is situated in the posterolateral angle of the body. The pereiopods all are of the same shape, the sixth pair being slightly smaller than the others. Each leg ends in a strong unguis and is provided at the base with an oostegite (= marsupial plate = incubatory lamella). The oostegite of the first leg (figs. 2*a*, *b*) not only covers the maxillipeds, but also the oostegite of the second leg. This first oostegite is very curious in that it consists of various lamellae. When seen in situ only the ventral lamella is visible. This lamella is about as long as broad, is of an irregular

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PLATE I

Fig. a. *Parapandalus narval* (Fabr.) parasitized by *Zonophryxus dodecapus* nov. spec. × 0.8.

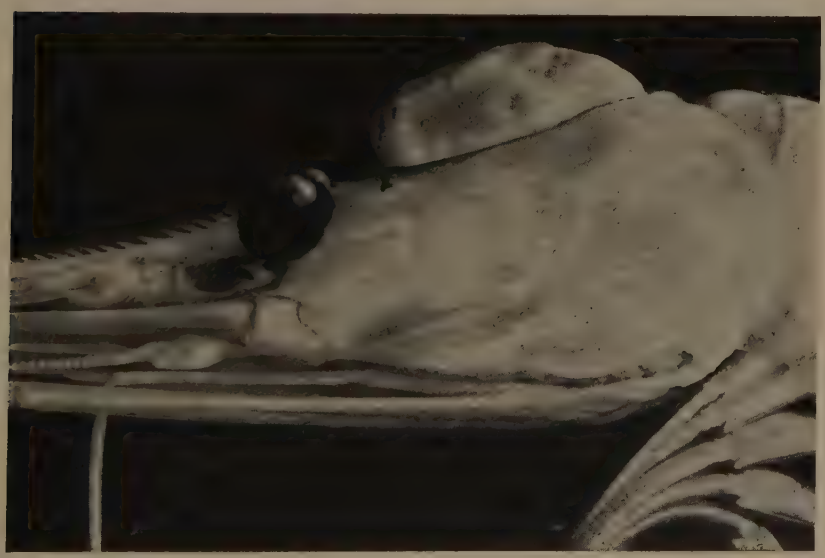

Fig. b. *Zonophryxus dodecapus* nov. spec. shown on carapace of *Parapandalus narval* (Fabr.). × 3.

outline and possesses a deep incision in the posterior part of the inner margin. A second lamella, which is about as broad as the first, is fused with its base to the first lamella along a line running transversely over the middle of the ventral surface of the first lamella. The second lamella

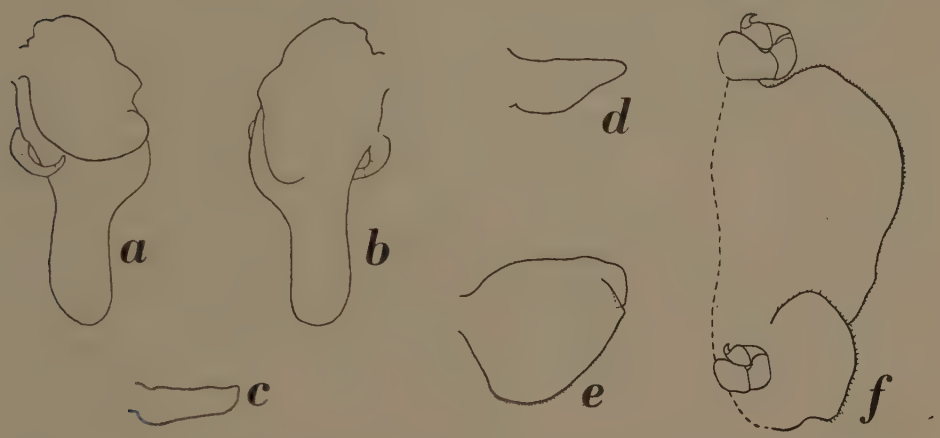

Fig. 2. *Zonophryxus dodecapus* nov. spec., female. *a*, oostegite of first pereiopod, ventral view; *b*, same, dorsal view; *c*, oostegite of second pereiopod; *d*, oostegite of third pereiopod; *e*, oostegite of fourth pereiopod; *f*, fifth and sixth pereiopods with their oostegites. *a*—*f*, × 10.

posteriorly reaches far beyond the first, its tip is tongue-like narrowed. A third, smaller, lamella is fastened to the basal part of the underside of the second. Near the base of the second lamella a palp-like process is visible; this palp curves over the lateral margin of the second lamella, and in it very faint indications of a possible segmentation may be seen. The nature of this palp, which is not mentioned in any of the other descriptions of species of *Zonophryxus*, is not known to me. The oostegite of the second pereiopod (fig. 2*c*) is elongate in shape, it is directed towards the middle of the body and is entirely covered by the first oostegite. The oostegite of the third pair of legs (fig. 2*d*) is slightly larger than that of the second, it is more triangular, and it is covered by the oostegites of the fourth and fifth legs. The fourth oostegite (fig. 2*e*) is broadly laminar, being slightly broader than long, it is almost completely covered by the fifth oostegite. The latter (fig. 2*f*) is very large: the oostegites of the fifth legs together cover the larger part of the posterior half of the ventral side of the body. The fifth oostegite gradually narrows posteriorly, it has the outer margin entirely fused with the body wall, while the posterior margin partly is fused with the oostegite of the sixth leg. The sixth leg is situated in the postero-lateral angle of the body and bears a rounded oostegite, which is fused partly with the lateral and posterior parts of the body wall and partly with the fifth oostegite. No pleopods are present. The three raised pads in the extreme posterior end of the ventral surface of the body mentioned by

BARNARD (1913, p. 229) for *Zonophryxus quinquedens,* are visible in my specimens too.

♂. The male (figs. 1*c, d*) has the body flattened. The head is broad, being slightly narrower than the thoracic segments. The upper surface of the head is somewhat concave. The first thoracic segment is fused with the cephalon. The other six thoracic segments are free and distinct, the posterior segments are somewhat narrower than the anteriors. The abdomen is unsegmented, it is elongate triangular, with the greatest breadth slightly posterior of the base and regularly tapering towards the top. The seven pereiopods are uniform in shape.

Remarks. The genus is characterized by the curious ventro-lateral border, which surrounds the entire ventral surface of the female, and which invariably bears 9 to 11 teeth on the posterior margin. At present there are six species known to belong to this genus: *Zonophryxus retrodens* Richardson (1904, p. 678, figs. 30—33; 1906, p. 823, figs. 4, 5) from Oahu, Hawaiian Islands, host unknown (type of the genus); *Zonophryxus grimaldii* Koehler (1911, p. 16, figs. 13, 14), from S. of the Gulf of Cadiz, Spain, depth 1401 m, host *Heterocarpus grimaldii* A. Milne Edwards & Bouvier; *Zonophryxus trilobus* Richardson (1910, p. 41, figs.) from Caluya Island, Philippines, depth 562 m, host unknown; *Zonophryxus quinquedens* Barnard (1913, p. 228, pl. 22), from off Cape Point, S. Africa, depth 850—1250 m, host *Nematocarcinus* spec.; *Zonophryxus similis* Richardson (1914, p. 369, fig. 16), Pacific Ocean S. of Panama, depth 1000 m, host unknown; and the present form,

The present species differs from all 5 already known species by possessing six instead of five pereiopods. Curiously enough in the drawings of the species of *Zonophryxus* species, nearly always the oostegites of the sixth pereiopods are visible; they are, however, generally considered to be pleopods. In almost all other genera of Dajidae the female is known to possess 5 or less pairs of pereiopods only. The only exception is the genus *Paradajus* Nierstrasz & Brender à Brandis (1923, p. 108, pl. 8 fig. 30), in which the female is provided with six legs; the only species of this genus, *Paradajus tenuis* Nierstr. & Br. à Br., is a parasite found in the branchial cavity of a Brachyuran crab from the Malay Archipelago, it differs considerably in many respects from *Zonophryxus*. The character of the presence of six pereiopods at first sight seems to be of generic importance. I think, however, the erection of a new genus for the present species not justified, since in all other characters it is a typical *Zonophryxus*.

Richardson in her descriptions gives rather few details of the antennae, mouth parts and oostegites. Her *Z. retrodens* and *Z. similis* seem to have the external antennae divided into three distinct joints. The maxillipeds mentioned by Richardson (1914) for *Z. similis,* probably are those organs, which are considered by BARNARD (1913) to be the lower lip.

KOEHLER (1911) describes the antennae, the oral parts and the oostegites of his specimen quite accurate, though he did not dissect his material. His

description of the various details agrees very good with the characters found in my material. KOEHLER states the external antenna to consist of two joints. The drawing he gives of it shows the situation much like in *Zonophryxus dodecapus*. *Zonophryxus grimaldii* at once may be distinguished from the other species of this genus by having the ventro-lateral border posteriorly ending in a narrow point.

BARNARD (1913) gives many details of the oral parts and the antennae of his species. The antennae agree good with those of the present form, as also do the mandibles. The maxillipeds, however, are quite different, while BARNARD moreover reports the presence of maxillae in one of his specimens. No maxillae could be found in my material.

At present the host of three species of *Zonophryxus* is known. In all three cases it belongs to the Crustacea Decapoda Caridea: *Heterocarpus grimaldii* A. M. Edw. & Bouv. and *Parapandalus narval* (Fabr.) both belong to the family Pandalidae, while *Nematocarcinus* belongs to the family Nematocarcinidae. Up till now it was not known how the parasite was attached to its host. The present material of *Zonophryxus dodecapus* shows that the parasites are attached to the posterodorsal region of the carapace of their hosts, just like the species of *Holophryxus*.

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Chemistry. — *Influence of temperature on the substitution type in the bromination and chlorination of aromatic compounds in the gas phase.* By F. L. J. SIXMA and J. P. WIBAUT.

(Communicated at the meeting of February 26, 1949.)

§ 1. The rules governing the introduction of substituents into the benzene nucleus during reactions in the liquid phase were determined by extensive investigations in the latter half of the nineteenth and the beginning of this century; in this respect the work of HOLLEMAN and his co-workers occupies a prominent place (1). Two cases can be distinguished:

a. the new substituent preferably occupies the ortho and para positions (ortho-para substitution);

b. the new substituent chiefly occupies the meta position (meta substitution).

To which of these two types the substitution reaction belongs was found to be in general independent of the nature of the new substituent and to be conditioned only by the substituent present already in the benzene nucleus. Investigations carried out by WIBAUT and co-workers, however, showed that when halogen atoms are introduced into aromatic compounds in the gas phase between 300° and 600°, the substitution type will in many cases be governed by the reaction temperature. Thus ortho and para dibromobenzene are the main products of the non-catalytic bromination of *monobromobenzene* (2,3) below 350° in the gas phase; in this case the reaction is therefore of the ortho-para type, which is also the case with the reaction in the liquid phase. Between 350° and 450°, however, a sudden change takes place; the quantity of *p*-dibromobenzene rapidly decreases, whereas the quantity of *m*-dibromobenzene rises; the relative quantity of *o*-dibromobenzene, on the contrary, remains almost unchanged. Upwards of 450° *m*-dibromobenzene is the main product of the reaction (fig. 1). According to HOLLEMAN's criterion the ortho-para substitution type therefore changes into the meta substitution type between 350° and 450°. A similar change was observed in the bromination of chlorobenzene (fig. 2) and of fluorobenzene [1]). In these experiments purified pumice and graphite were used as contact material.

If the bromination reaction is carried out by means of a ferric bromide (or ferric chloride) catalyst on a suitable carrier, the total reaction velocity is much greater. In the temperature range from 200—450° the substitution reaction develops according to the ortho-para type; the substitution type

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[1]) Owing to experimental difficulties the quantities of ortho and meta fluorobenzene in the reaction product of the bromination of fluorobenzene could not be determined.

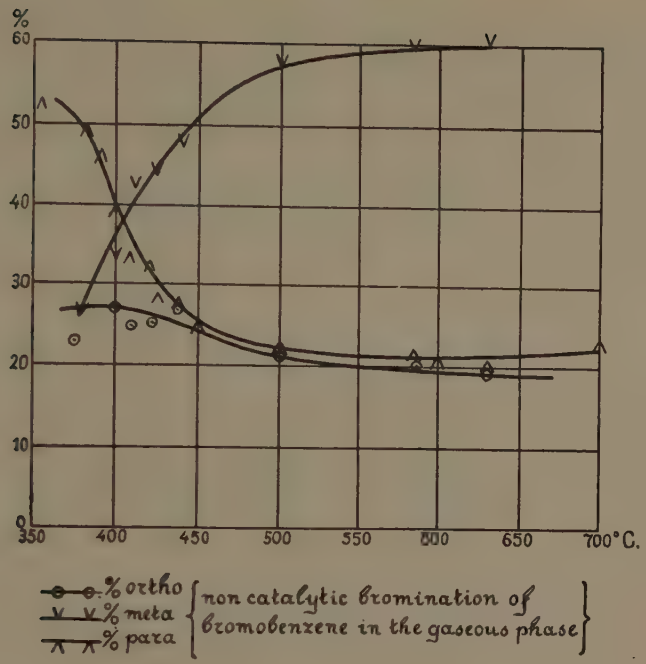

%
60
50
40
30
20
10
0
350
400
450
500
550
600
650
700° C.
% ortho
% meta
% para
non catalytic bromination of bromobenzene in the gaseous phase

Fig. 1.

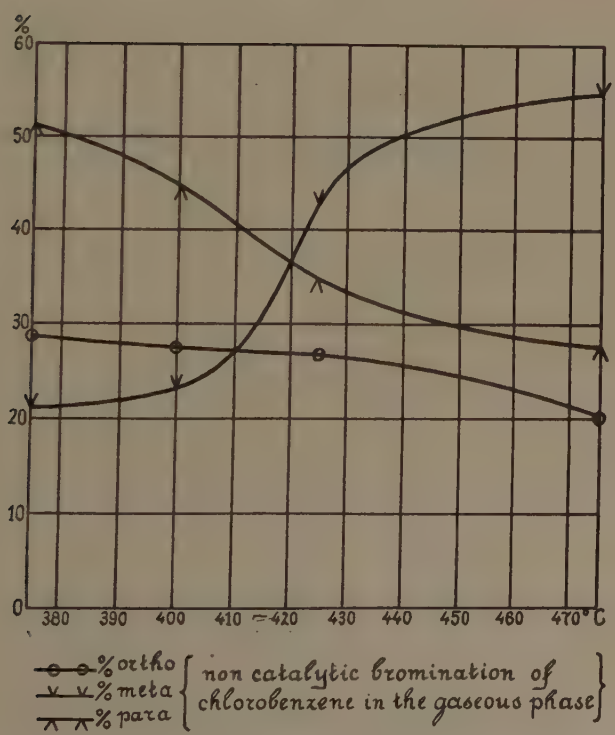

%
60
50
40
30
20
10
0
380
390
400
410
420
430
440
450
460
470° C
% ortho
% meta
% para
non catalytic bromination of chlorobenzene in the gaseous phase

Fig. 2.

does not change. Experimental difficulties prevented the investigation of this heterogeneously catalytic reaction upwards of 450°.

The ratio of the isomeric dibromobenzenes formed in the catalytic bromination of bromobenzene is represented by the following formulae of SCHEFFER (4):

$$ln \frac{2 \text{ conc.}_p}{\text{conc.}_m} = \frac{E_m - E_p}{RT} \quad \ldots\ldots\ldots \quad (1)$$

and

$$ln \frac{2 \text{ conc.}_p}{\text{conc.}_o} = \frac{E_o - E_p}{RT} \quad \ldots\ldots\ldots \quad (2)$$

assuming that $E_m - E_p = 2890$ cal/mol and $E_o - E_p = 2036$ cal/mol; E_o, E_m and E_p in these formulae represent the energies of activation of the substitution in the ortho, meta and para positions [2]). Fig. 3 shows satisfactory agreement between the values calculated according to formulae

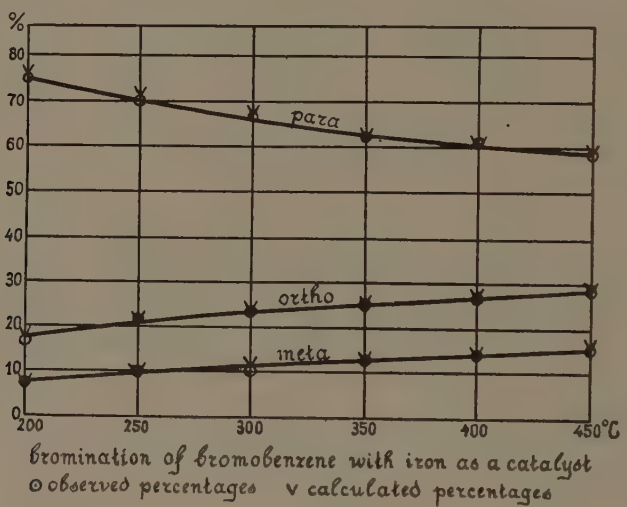


Fig. 3.

(1) and (2) and the experimental figures. This investigation carried out in 1937 by WIBAUT and VAN LOON (2) therefore showed that the formulae of SCHEFFER hold within a temperature range of 250°. Before that time these formulae had been tested for a narrow temperature range (—30° to ca + 30°) on a number of nitration reactions.

The results of the non-catalytic bromination of the halogenobenzenes (figs. 1 and 2) cannot be represented by formulae of the types (1) and (2). Neither can the more general formulae derived by SCHEFFER, which contain an energy term as well as an entropy term account for the change

[2]) In our former publication the energy of activation was represented by ε; for typographical reasons the symbol E is used in this paper.

in the para:meta ratio which takes place within a narrow temperature range. To gain a better insight into the influence of the temperature on substitution reactions in aromatic rings, the monobromination of naphthalene has been investigated extensively in our laboratory (6, 8, 9). Introduction of one substituent into the naphthalene molecule is theoretically simpler than into a mono-substituted benzene (C_6H_5A), because in the former case only two isomers are formed. Moreover, the naphthalene molecule consists only of carbon and hydrogen atoms, whereas the substituent in a monosubstituted benzene has in most cases a polar character, which enables it to influence the charge distribution in the molecule.

In our former publication (9) we subjected the results obtained in the bromination of naphthalene to a theoretical discussion, special attention being paid to the reversible conversion α-bromonaphthalene $\rightleftarrows$ β-bromonaphthalene, which takes place under the influence of catalysts such as ferric chloride or ferric bromide. We shall now consider the *non-catalytic bromination* of naphthalene, which is not accompanied by conversion of the monobromonaphthalenes into each other — neither at elevated temperatures nor otherwise —, which has been proved experimentally.

The monobromination of gaseous naphthalene in the presence of a non specific contact material (pumice, glass wool) yields in the temperature range from 250°—300° chiefly α-bromonaphthalene in addition to small quantities of β-bromonaphthalene. In this temperature range the $\alpha : \beta$ ratio satisfies SCHEFFER's formula and is therefore determined by the difference in energies of activation required for the substitution in the α- and β-positions. In the temperature range from 500—650° equal quantities of α- and β-bromonaphthalene are formed; the difference in energy of activation does no longer play a part and the $\alpha : \beta$ ratio is only determined by the probability of the collision of a bromine particle with the naphthalene molecule in the α- or β-position.

In the transition zone from 350—500° the relative quantity of β-isomer rapidly increases, which is reflected in the *S*-shape of the curve (see fig. 4). The question is how to account for the experimental result that from a certain temperature the difference in energy of activation no longer plays a part and that within a narrow temperature range a rather sudden change in the substitution type develops.

In our first paper reference was made to a theoretical consideration by one of us (*S.*), on the strength of which the existence of a transition temperature was made plausible (9). It was found, however, that the formulae derived lead to a less sudden change in the substitution type than had been observed experimentally.

We shall now derive a formula by means of which the *S*-shaped curve found experimentally can be represented.

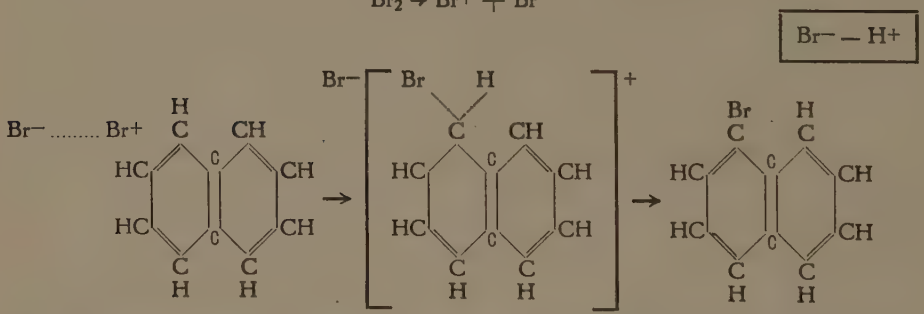
§ 2. In the bromination of gaseous naphthalene we must distinguish between a wall reaction and a real gas reaction. At a low temperature the

former will predominate (5). Our investigations into the isomerization of bromonaphthalene in the gas phase in the presence of a ferric bromide contact indicate that upwards of 400° the adsorption of the bromonaphthalene molecules to the catalyst surface is very slight. Obviously, we may assume that the adsorption of naphthalene on any solid surface at that temperature will also be negligible, so that we may put forward the hypothesis that the bromination of naphthalene vapour at about 300° and below will chiefly proceed as a wall reaction and from 400° upwards chiefly as a gas reaction.

We shall now explain how the change from wall reaction to gas reaction may lead to a pronounced change in the ratio of the isomeric substitution products. In our former publication we showed that the monobromination of liquid naphthalene, which takes place in the temperature range from 85—215°, can be described as an electrophylic substitution. By starting from this theory and using wave-mechanical calculations, one of us (*S.*) found an approximate value for the difference in energy of activation for substitution in the α- and β-positions ($E_\beta - E_\alpha$); this theoretical value was in fair agreement with the value calculated from the experimental data by means of SCHEFFER's formula (10).

We now assume that the bromination in the gas phase in the temperature range from 250—300° also takes place according to an electrophylic substitution type. For this it is necessary that a bromine particle is strongly polarized in the transition state of the reaction, which can be represented schematically by a splitting into a positive and a negative bromine ion:

$$Br_2 \rightarrow Br^+ + Br^-$$



As stated formerly, catalysts such as ferric bromide, aluminium chloride or iodine can stabilize the transition state by the formation of complexes with the negative bromine particle:

$$Br^- + FeBr_3 \rightarrow [FeBr_4]^- \quad . \; . \quad (3) \qquad Br^- + I_2 \rightarrow [BrI_2]^- \quad . \; . \quad (4)$$

This increases the reaction velocity.

Such a stabilization of the transition state by a second bromine molecule is also conceivable:

$$2\,Br_2 \rightarrow Br^+ + [Br_3]^- . \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

This is in agreement with the fact that some brominations in the liquid

phase proceed according to a second order reaction with respect to the bromine concentration (11).

In our experiments on the non-catalytic bromination in the absence of metal halides, the stabilization of the transition state may proceed according to equation (5). It should be borne in mind, however, that collisions between three molecules in a gas phase reaction are rare. In the temperature range from 250—300°, however, the reaction will chiefly proceed on the wall, so that in this range the electrophylic reaction is possible. According as the temperature rises, the adsorption of the reacting substances to the wall decreases; the normal rise in reaction velocity with the temperature is thus counteracted. The electrophylic substitution reaction on the wall is ultimately reduced to such an extent that the reaction in the gas phase predominates. For this reaction we assume a radical mechanism. As a result of the thermal dissociation of the bromine molecules bromine atoms are formed, which may introduce a chain reaction. This does not require adsorption to the wall. The chain reaction is represented by the following scheme:

$$\mathrm{Br_2} \underset{K_2}{\overset{K_1}{\rightleftarrows}} 2\,\mathrm{Br}\cdot$$

$$\mathrm{Br}\cdot + \mathrm{C_{10}H_8} \overset{K_{r\alpha}}{\underset{K_{r\beta}}{\rightleftarrows}} \begin{cases} \alpha\text{—}\mathrm{C_{10}H_7}\cdot + \mathrm{HBr} \\ \beta\text{—}\mathrm{C_{10}H_7}\cdot + \mathrm{HBr} \end{cases}$$

$$\alpha\text{—}\mathrm{C_{10}H_7}\cdot + \mathrm{Br_2} \xrightarrow{K'_{r\alpha}} \alpha\text{—}\mathrm{C_{10}H_7Br} + \mathrm{Br}\cdot$$

$$\beta\text{—}\mathrm{C_{10}H_7}\cdot + \mathrm{Br_2} \xrightarrow{K'_{r\beta}} \beta\text{—}\mathrm{C_{10}H_7Br} + \mathrm{Br}\cdot$$

Bij means of the values of bond energies mentioned by PAULING (12) the heat effect of these reactions can be computed. The reactions $K_{r\alpha}$ and $K_{r\beta}$ will show hardly any heat effect, while in the reactions $K'_{r\alpha}$ and $K'_{r\beta}$ about 8 kcal/mol of energy is liberated. It is therefore conceivable that the reactions should proceed according to this scheme. The chains can be broken by:

$$2\,\mathrm{Br}\cdot \xrightarrow[K_2]{} \mathrm{Br_2}.$$

which reaction may proceed at the wall of the vessel.

At low temperatures the overall rate of this radical substitution will be low, because the dissociation of the bromine is then extremely slight. According as the temperature rises, the dissociation increases (at 800° 0.16 % of the bromine has been dissociated; at 900° 1.5 %); the rate of the atomic reaction rises, whereas the rate of the electrophylic reaction decreases. At a certain temperature the atomic substitution will therefore predominate.

§ 3. To obtain a quantitative formulation we assume that the dissociation equilibrium of the bromine is established so quickly that the concentration of the bromine atoms during the experiment may be considered constant. If the heat of dissociation of bromine is D cal/mol and C_D the corresponding entropy term, the concentration of the bromine atoms is:

$$[\mathrm{Br}\cdot] = C_D^{1/2} \cdot e^{-D/2RT} \cdot [\mathrm{Br_2}]^{1/2}.$$

The rate at which α- and β-monobromonaphthalene are formed according to the radical reaction is then:

$$\left(\frac{d\alpha}{dt}\right)_r = C_D^{1/2} \cdot C_{r\alpha} \cdot e^{(-E_{r\alpha} - 1/2 D)/RT} \cdot [\mathrm{C_{10}H_8}] \cdot [\mathrm{Br_2}]^{1/2} \quad . \quad . \quad . \quad (6)$$

$$\left(\frac{d\beta}{dt}\right)_r = C_D^{1/2} \cdot C_{r\beta} \cdot e^{(-E_{r\beta} - 1/2 D)/RT} \cdot [\mathrm{C_{10}H_8}] \cdot [\mathrm{Br_2}]^{1/2} \quad . \quad . \quad . \quad (7)$$

In these equations $E_{r\alpha}$ and $E_{r\beta}$ are the energies of activation for the reactions $K_{r\alpha}$ and $K_{r\beta}$; $C_{r\alpha}$ and $C_{r\beta}$ are the corresponding collision factors, whose dependence on the temperature is relatively small. Since each naphthalene molecule has as many α- as β-positions and both positions are sterically equally favoured, we put $C_{r\alpha} = C_{r\beta}$.

The electrophylic reaction, which develops exclusively on the wall, is based on adsorption. In the case of catalytically inactive adsorbents such as pumice or glass wool, we have adsorption by VAN DER WAALS forces; the number of adsorbed molecules is proportional to $e^{E_{ads}/RT}$, where E_{ads} is a function of the heat of adsorption of naphthalene, bromine and the reaction products. The rate at which α- and β-monobromonaphthalene are formed according to the electrophylic reaction is then:

$$\left(\frac{d\alpha}{dt}\right)_e = C_{e\alpha} \cdot C_{ads} \cdot e^{(-E_{e\alpha} + E_{ads})/RT} \cdot [\mathrm{C_{10}H_8}] \cdot [\mathrm{Br_2}]^n \quad . \quad . \quad . \quad (8)$$

$$\left(\frac{d\beta}{dt}\right)_e = C_{e\beta} \cdot C_{ads} \cdot e^{(-E_{e\beta} + E_{ads})/RT} \cdot [\mathrm{C_{10}H_8}] \cdot [\mathrm{Br_2}]^n \quad . \quad . \quad . \quad (9)$$

In these equations the bromine concentration occurs to the power n, n being 1 or 2 according as the reaction mainly proceeds with two or three collisions (cf. § 2). The collision factors $C_{e\alpha}$ and $C_{e\beta}$ are now equalized again; also E_{ads} has the same value in both equations, because of the simultaneous reactions between the adsorbed bromine and naphthalene molecules. $E_{e\alpha}$ and $E_{e\beta}$ are the energies of activation for the electrophylic substitution reactions in the α- and β-positions in the naphthalene nucleus. The ratio in which α- and β-monobromonaphthalene will occur in the reaction product is:

$$\frac{[\alpha]}{[\beta]} = \frac{\left(\frac{d\alpha}{dt}\right)_e + \left(\frac{d\alpha}{dt}\right)_r}{\left(\frac{d\beta}{dt}\right)_e + \left(\frac{d\beta}{dt}\right)_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

After substitution of (6)—(9) in (10) and dividing numerator and denominator by $\left(\frac{d\beta}{dt}\right)_e$ we find:

$$\frac{[\alpha]}{[\beta]} = \frac{e^{(E_{e\beta}-E_{e\alpha})/RT} + C \,.\, e^{Q_\alpha/RT}}{1 + C \,.\, e^{Q_\beta/RT}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where

$$C = \frac{C_D^{\frac{1}{2}} \,.\, C_{r\alpha} \,.\, [Br_2]^{\frac{1}{2}-n}}{C_{e\beta} \,.\, C_{ads}} = \frac{C_D^{\frac{1}{2}} \,.\, C_{r\beta} \,.\, [Br_2]^{\frac{1}{2}-n}}{C_{e\beta} \,.\, C_{ads}}$$

and

$$Q_\alpha = + E_{e\beta} - \tfrac{1}{2} D - E_{ads} - E_{r\alpha};$$
$$Q_\beta = + E_{e\beta} - \tfrac{1}{2} D - E_{ads} - E_{r\beta}.$$

As the bromination of naphthalene below 100° in the liquid phase proceeds fairly rapidly, $E_{e\alpha}$ and $E_{e\beta}$ will be of the order of 10 to 20 kcal/mol. The dissociation heat (D) of bromine is about 46 kcal/mol, so that Q_α and Q_β are negative magnitudes. From theoretical considerations as well as from the experiment we know that the difference between $E_{e\beta}$ and $E_{e\alpha}$ is a positive magnitude (about 4 kcal/mol) (9, 10). At elevated temperatures the terms $C.\, e^{Q/RT}$ will therefore predominate in equation (11).

The experiment showed that at elevated temperatures α- and β-bromonaphthalene are formed in equal quantities (§ 1), so that $\frac{[\alpha]}{[\beta]} = 1$. This holds if $Q_\alpha = Q_\beta$. The absolute values of $E_{r\alpha}$ and $E_{r\beta}$ will be much lower than those of $E_{e\alpha}$ or $E_{e\beta}$. This renders it possible that even at moderate temperatures the average energy of the molecules is so great that any sterically favourable collision of a bromine atom with a naphthalene molecule may cause a reaction. The orienting effect is then eliminated and $E_{r\alpha}$ and $E_{r\beta}$ can be equalized, so that

$$\frac{[\alpha]}{[\beta]} = \frac{C \,.\, e^{Q_\alpha/RT}}{C \,.\, e^{Q_\beta/RT}} = 1.$$

On the other hand, $C.e^{Q/RT}$ will be very small at a low temperature in the gas phase, so that in this case

$$\frac{[\alpha]}{[\beta]} = e^{(E_{e\beta}-E_{e\alpha})/RT}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The electrophylic reaction then predominates and SCHEFFER's relation is found again. For the bromination of naphthalene at a low temperature in the gas phase $E_{e\beta} - E_{e\alpha} = 4215$ cal/mol.

§ 4. Unfortunately, there are no data available to calculate Q and C independently of the results of the bromination experiments. Therefore we can only calculate these magnitudes from the ratio in which the isomeric substitution products are formed at two different temperatures. The

difference in energy of activation for the electrophylic substitution ($E_{e\beta}$—$E_{e\alpha}$) in the non-catalytic bromination of naphthalene can be calculated from the ratio in which α- and β-isomers are formed at a temperature below the transition zone (6).

By means of the following values of the constants we find satisfactory agreement between the calculated and experimental values for the bromination and for the chlorination of naphthalene (see table I and II and fig. 4).

	Q cal/mol	$E_{e\beta}$—$E_{e\alpha}$ cal/mol	C
bromination of naphthalene	-3.8×10^4	4215	5×10^{12}
chlorination of naphthalene	-2.8×10^4	(4215)	3×10^{12}

For the chlorination of naphthalene ($E_{e\beta}$—$E_{e\alpha}$) cannot be calculated according to the above method, because the measurements cannot be extended to temperatures sufficiently below the transition zone (cf. fig. 4);

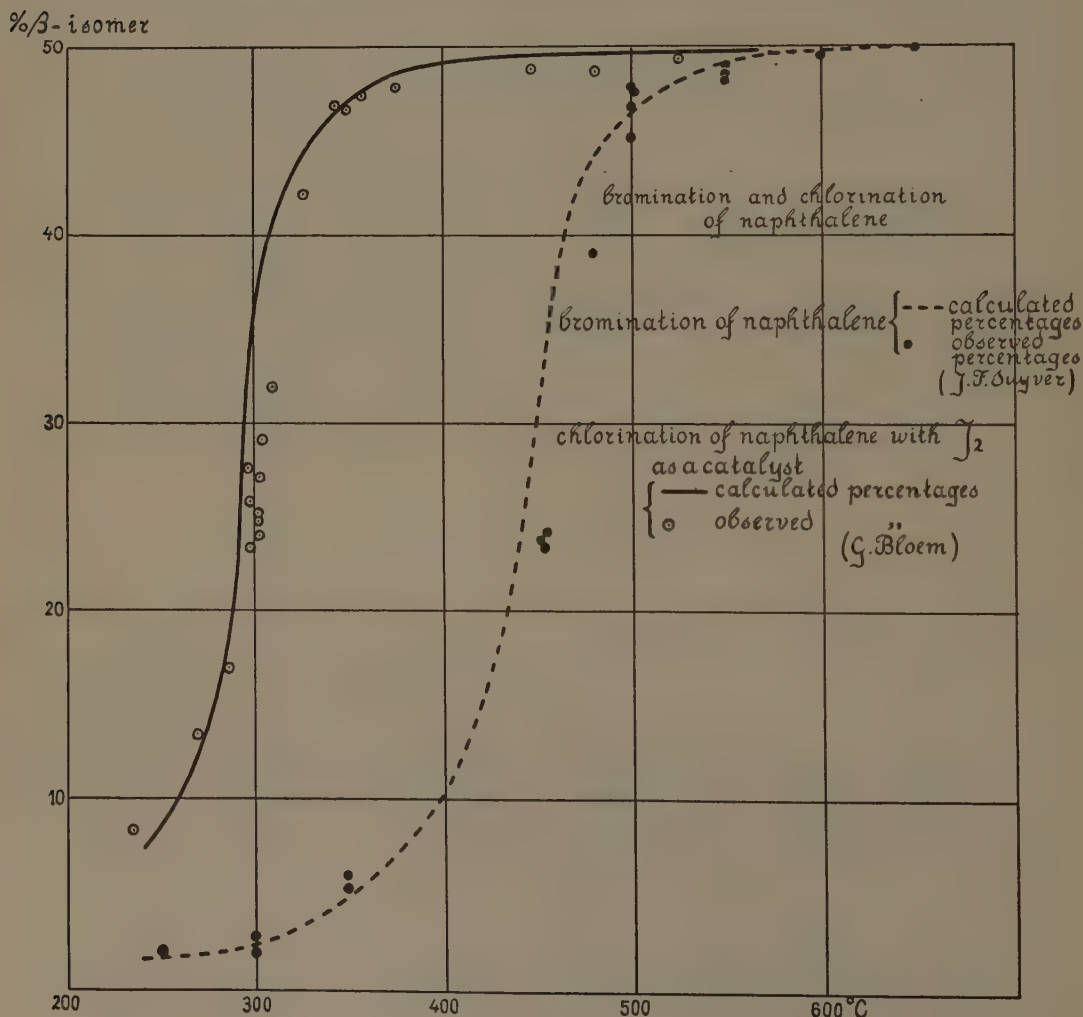


Fig. 4.

we assumed the same value as found for the bromination of naphthalene, i.e. 4215 cal/mol.

TABLE I.

Non-catalytic bromination of naphthalene in the gas phase (6).

Temperature ° C	% β-monobromonaphthalene	
	experimental (J. F. SUYVER)	calculated
250	1.7	1.7
300	2.4	2.4
350	5.5	4.9
450	23.9	32.4
500	46.7	46.5
550	49.0	49.7
650	50.0	50.0

TABLE II.

Chlorination of naphthalene in the gas phase (under the influence of iodine) (7)

Temperature ° C	% β-chloronaphthalene	
	experimental (G. BLOEM)	calculated
250	10.4	8.7
300	24—29	36.0
350	46.9	47.5
450	48.9	49.9
550	49.9	50.0

In the steepest part of the curve the deviation between the experimental and the calculated figures is fairly great. As the ratio of the isomers varies considerably with the temperature in this range, both the calculated and the experimental results are less accurate.

The ratio in which α- and β-bromonaphthalene are formed in the bromination of naphthalene in the gas phase with iron compounds as catalysts cannot be represented either by our own formula or by SCHEFFER's. As we showed in a previous publication, the two isomers are converted into each other under these conditions (9).

In the bromination of bromobenzene over iron compounds as catalyst such a conversion of the isomeric reaction products does not take place (2, 10), so that for this case formula (11) can be applied. These iron compounds, however, only catalyze the electrophylic reactions, the rate of the radical substitution reaction remaining unchanged. The result is that in the catalytic bromination of bromobenzene the electrophylic wall reaction will predominate up to 450°; the sudden change in the substitution type is thus completely suppressed (fig. 3). In formula (11) the terms $C.e^{Q/RT}$

are therefore smaller than exp $(E_{e\,\text{meta}} - E_{e\,\text{para}})/RT$ so that this relation can be replaced by SCHEFFER's formula (form. 12; cf. § 1).

In the non-catalytic bromination of bromobenzene the substitution type changes. The ratio in which para and meta dibromobenzene are formed is then represented by formula (11). The ortho position is partly screened by steric influences, so that

$$C_{e\,\text{ortho}} \neq C_{e\,\text{para}} \quad \text{and}$$
$$C_{r\,\text{ortho}} \neq C_{r\,\text{para}}.$$

As the magnitude of this effect is unknown we shall restrict the calculation to the ratio meta : para dibromobenzene. The difference in energy of activation for the electrophylic reaction $(E_{e\,\text{meta}} - E_{e\,\text{para}})$ is derived from the values found for the catalytic bromination. In this calculation we assume therefore that the catalyst reduces the energies of activation for the meta and para substitution reaction by the same amount, which supposition may hold only approximately.

In table III the experimental values for the para : meta isomer ratio in the non-catalytic bromination of bromobenzene are compared with the values derived from formula (11) with

$$Q = -5.65 \times 10^4 \text{ cal/mol}$$
$$E_{e\,\text{meta}} - E_{e\,\text{para}} = 2890 \text{ cal/mol}$$
$$C = 9.6 \times 10^{18}.$$

TABLE III.

non-catalytic bromination of bromobenzene in the gas phase (2).

Temperature °C	Ratio of the reaction velocities $\frac{K_{para}}{K_{meta}}$ [1]	
	experimental (V. LOON)	calculated
380	4.18	4.18
400	2.45	2.41
410	1.64	1.685
420	1.51	1.35
440	1.135	1.11

Experiments are in progress to test the hypothesis underlying this theory of the influence of the temperature on the substitution type, i.e. the development of a radical type of reaction at high temperatures.

January 1949.

Laboratory of Organic Chemistry of the University of Amsterdam.

[1]) Calculation of percentages was impossiblee in this case, because the reaction mixture contains the ortho isomer as third component.

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Physics. — *Unitary Quantum Electron Dynamics* II. By H. J. GROENEWOLD.
(Koninklijk Nederlands Meteorologisch Instituut te De Bilt.)
(Communicated by Prof. F. A. VENING MEINESZ.)

(Communicated at the meeting of January 29, 1949.)

3. Unitary theory.

3.1 *Elimination of photons.* The equations of motion (36) show that an empty state cannot subsist in time. Photons are continually emitted and absorbed. That can still more clearly be seen from (37) where in each exponential corresponding to an infinitesimal time interval $t_k^{(m_k)}\, t_k^{(m_k+1)}$ the creation and annihilation operators describe (virtual or actual) emission and absorption of photons.

In spite of this we suppose for a moment that at a certain set of initial times $(t_1', \ldots t_n')$ the wave function is an empty state function and that at a set of final times $(t_1, \ldots t_n)$ it is again an empty state function. Or expressed less correctly no photons are supposed to be present at the initial and final electron times. In that case all photons emitted in the meantime are absorbed in the meantime and vice versa. These transient photons can be eliminated from the description indeed by dovetailing according to 2.183 the empty-empty part of the entire operator in (37) which transforms the initial wave function into the final one. Doing so we get

$$\left.\begin{array}{l}(x_1, \ldots x_n \mid \Psi = \lim \prod_{k=1}^{n} \prod_{m=1}^{p} \left(\int d\vec{x}_k^{(m_k)}\right) \mathbf{N}_k\left(x_k^{(m_k+1)}; x_k^{(m_k)}\right) \\ e^{\frac{ie^2}{\hbar c}\frac{1}{2}\sum_{k,l=1}^{n}\left(x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}\right)\left(x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'}\right) W_{\alpha\alpha'}\left(x_k^{(m'_k)}, x_l^{(m'_l)}\right)} \\ (x_1', \ldots x_n' \mid \Psi.\end{array}\right\} \quad (38)$$

Now the underlying supposition is untenable because all photons emitted by the electrons during the time intervals $t_k'\, t_k$ $(k = 1, \ldots n)$ cannot all be absorbed by other electrons during these time intervals and vice versa. That might suggest to solve the difficulty by shifting the initial and final times toward the infinite past and future. But that would be too much. For whereas it is doubtful whether the difficulties would not be merely shifted toward infinity, it is sure that all transitory adventures of the electrons would be shifted out of the picture. Now that not all electrons can simultaneously be saved from this deadly eternity, no much else seems to be left than trying to save them one at a time. So each electron will in its turn be taken separately with an arbitrarily variable finite initial and final time and the initial and final times of all other electrons are shifted toward the infinite past and future. That makes us sure that if the opaqueness con-

dition is fulfilled (which may require an infinite n) all photons emitted or absorbed by the chosen electron during the finite interval of time are absorbed or emitted by the other electrons during the infinite intervals of time. But we are not sure whether all other difficulties have been solved. Thus we cannot say that the photons have rigorously been eliminated from the description.

3.2 *Equations of motion.* Therefore we take another way. We forget everything about dualistic theory and try what happens in a unitary theory with equations of motion

$$\left.\begin{aligned} &(x_1, \ldots x_n \mid \Psi_{t_1=\ldots t_{r-1}=t_{r+1}=\ldots t_n \to +\infty} \\ &\qquad = \lim \prod_{k=1}^{n} \prod_{m=1}^{p} \left(\int (d\vec{x}_k^{(m_k)})\, \mathbf{N}_k\, (x_k^{(m_k+1)}; x_k^{(m_k)})\right. \\ &e^{\frac{ie^2}{\hbar c}\frac{1}{2}\sum_{k,l=1}^{n}\sum_{m=1}^{p} (x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha})\,(x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'})\, W_{\alpha\alpha'}\,(x_k^{(m'_k)}, x_l^{(m'_l)})} \\ &(x'_1, \ldots x'_n \mid \Psi_{t'_1=\ldots t'_{r-1}=t'_{r+1}=\ldots t'_n \to -\infty} \qquad (r=1, \ldots n). \end{aligned}\right\} \quad (39)$$

If a unitary theory of electron interaction cannot be like this it is hard to conceive whatelse it could be like.

3.3 *Consistency and applicability.* After having written down the equations (39) one is bound to state whether and under which conditions they form a consistent system. Further one is expected to give the conditions under which they are able to describe the processes, which are described by the dualistic theory. Finally after having established a complete self supporting unitary description one should be able to derive the dualistic theory by introducing in the formalism of the unitary theory wave operators of (from this point of view) fictitious photons. It is needless to say that we have written down (39) without being able to meet these obligations. We can do no more than making some comments on the general principles and illustrating the unitary description of some fundamental radiation processes in a very simple model. The comments are given in the present paper **2**, the illustration will be given in a later paper **3**.

3.31 *Boundary conditions.* One of the most difficult problems of consistency is that of boundary conditions. In 4-dimensional time-space an intricate network of mutual interaction is interwoven between the electrons. It looks like flogging a dead horse to disentangle this network in order to indicate in general initial conditions, starting from which the further course of the system can be derived. This is the main reason why in applications we have to resort to oversimplified models.

3.32 *Supplementary conditions.* It has already been announced in 2.23 that in dualistic theory one needs supplementary conditions. Therefore we have to be prepared that also in unitary theory one might need supplementary conditions. We shall see, however, in 4.33 that presumably one does not.

3.33 *Opaqueness condition.* From the dualistic point of view it has

already been remarked in 1.4 that the opaqueness condition (there stated in dualistic terms) is necessary for the equivalence between dualistic and unitary description if the latter is possible at all. From the unitary point of view we shall see in the simple model treated in 3 that the opaqueness condition (then stated in unitary terms) is even necessary for the consistency of the unitary description.

3.34 *Relativistic invariance.* The form of the dualistic and unitary integral equations of motion (37) and (39) is not relativistic invariant. Because, however, the dualistic differential equations of motion (36) (multiplied by β_k) are relativistic invariant, (37) describes invariant processes. The notion of empty states is also invariant. Therefore if (39) describes any processes at all, they may be expected to be relativistic invariant as well. An obvious invariant formulation [2]) would be desirable.

3.35 *Independence of* j^a. Meanwhile the choice of the time-like vector j^a in (6) is a disguised form of fixing the direction of the time axis. Therefore in order to be genuinely relativistic invariant the theory has to be independent of the choice of j^a. Now in the dualistic description j^a only appears in the definition (22) of the interaction function $W_{\alpha\alpha'}$ and there even only in the cases *Dh* and *Hh*. The cases *Dd* and *Hd*, however, do lead to a description, which is independent of the choice of j^a.

3.36 *Asymmetry conditions.* In 3 we shall see that the asymmetry mentioned in 1.42 requires the occurrence of D_+ rather than D_a in the interaction function (22). That rules out the cases *Dd* and *Dh* for the purpose of unification.

3.4 Self interaction. The self terms ($k = l$) in the exponent in (39) ought to describe the self interaction of the electrons. They lead, however, to divergencies. These divergencies are provisionally removed by omitting the self terms. The relevant effects of self interaction are then supposed to be partially included in the unperturbed motion and therefore represented in the N's. Meanwhile entirely similar divergencies arise from the cross terms ($k \neq l$) in case of pair creation and annihilation. It belongs to Q_3 to find a modification which removes all divergencies and which properly deals with self interaction and with pair creation and annihilation.

4. Radiative and COULOMB interaction.

Now we compare the separation of the photon states in the dualistic description and of the interaction function in the unitary description into a transverse and a non-transverse part. With the help of the time-like vector j^a this separation can be made in a relativistic invariant way. But then it depends on the choice of j^a.

4.1 *Dualistic theory.* 4.11 *Scalar, longitudinal and transverse states.* The photon states belonging to the 4 wave functions $(y \mid {}^a\varphi \mid \vec{\xi}_\pm r)$ $(r = 1, 2, 3, 4)$ of (5) can be distinguished in to 1 scalar state $(r = 1)$,

1 longitudinal state ($r = 2$) and 2 transverse states ($r = 3, 4$) by chosing for the 1st and 2nd of the 4 4-vectors ${}^{\alpha}b$

$$\left.\begin{aligned} {}^{\alpha}b\,|\,\vec{\xi}_{\pm}1) &= j^{\alpha}, \\ {}^{\alpha}b\,|\,\vec{\xi}_{\pm}2) &= j^{\alpha} + \frac{\xi^{\alpha}}{j_{\gamma}\xi^{\gamma}}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (40)$$

where $\xi^0 = \pm\,|\,\vec{\xi}\,|$, and meanwhile observing (6). The scalar state vectors ($r = 1$) are the negative eigenvectors of $(g^{\alpha\beta} + 2j^{\alpha}j^{\beta})$ (cf. 2.141 and 2.151). The second relation of (6) can be split up into

$$\left.\begin{aligned} {}^{\alpha}b\,|\,\vec{\xi}_{\pm}1)(\vec{\xi}_{\pm}1\,|\,b^{\dagger}_{\beta} &= j^{\alpha}j_{\beta}, \\ {}^{\alpha}b\,|\,\vec{\xi}_{\pm}2)(\vec{\xi}_{\pm}2\,|\,b^{\dagger}_{\beta'} &= (j^{\alpha} + \frac{\xi^{\alpha}}{j_{\gamma}\xi^{\gamma}})(j_{\beta} + \frac{\xi_{\beta}}{j_{\delta}\xi^{\delta}}), \\ \sum_{r=3,4} {}^{\alpha}b\,|\,\vec{\xi}_{\pm}r)(\vec{\xi}_{\pm}r\,|\,b^{\dagger}_{\beta} &= \delta^{\alpha}_{\beta} - \frac{j^{\alpha}\xi_{\beta} + \xi^{\alpha}j_{\beta}}{j_{\gamma}\xi^{\gamma}} - \frac{\xi^{\alpha}\xi_{\beta}}{(j_{\gamma}\xi^{\gamma})^2}. \end{aligned}\right\} \quad . \quad (41)$$

The wave operators ${}^{\alpha}\varphi$ and $\varphi^{\dagger}_{\alpha}$ can be split up into a scalar, longitudinal and transverse part by taking apart in (8) the terms with $r = 1$, $r = 2$ and $r = 3, 4$

$$ {}^{\alpha}\varphi = {}^{\alpha}\varphi_{sc} + {}^{\alpha}\varphi_{ln} + {}^{\alpha}\varphi_{tr} \quad \ldots\ldots \quad (42)$$

and similarly for $\varphi^{\dagger}_{\alpha}$. Observing that $-i\hbar c\,\partial / \partial x^{\alpha}$ operating under the integral sign in (8) gives a factor ξ_{α} we can symbolically write

$$\left.\begin{aligned} {}^{\alpha}\varphi_{sc} &= -j^{\alpha}j_{\beta}\,{}^{\beta}\varphi, \\ {}^{\alpha}\varphi_{ln} &= (j^{\alpha} + \frac{\partial/\partial x_{\alpha}}{j_{\gamma}\partial/\partial x_{\gamma}})(j_{\beta} + \frac{\partial/\partial x_{\beta}}{j_{\delta}\partial/\partial x_{\delta}})\,{}^{\beta}\varphi, \\ {}^{\alpha}\varphi_{tr} &= (\delta^{\alpha}_{\beta} - \frac{j^{\alpha}\partial/\partial x^{\beta} + j_{\beta}\partial/\partial x_{\alpha}}{j_{\gamma}\partial/\partial x_{\gamma}} - \frac{\partial^2/\partial x_{\alpha}\partial x^{\beta}}{j_{\delta}\partial/\partial x_{\delta})^2})\,{}^{\beta}\varphi \end{aligned}\right\} \quad . \quad . \quad (43)$$

and similarly for $\varphi^{\dagger}_{\alpha}$. The separate parts satisfy the commutation relations

$$\left.\begin{aligned} &[(x\,|\,{}_{\alpha}\varphi'_{\beta sc}\,|\,y_0\},\{y_0\,|\,{}^{\beta}\varphi'^{\dagger}_{\alpha sc}\,|\,x')]^{-} = \\ &\qquad = (j_{\alpha}j_{\alpha'})(j^{\beta}j^{\beta'})\,G_{a\beta\beta'}(x-x'), \\ &[(x\,|\,{}_{\alpha}\varphi'_{\beta ln}\,|\,y_0\},\{y_0\,|\,{}^{\beta}\varphi'^{\dagger}_{\alpha ln}\,|\,x')]^{-} = \\ &\qquad = (\delta^{\beta}_{\alpha} + j_{\alpha}j^{\beta})(\delta^{\beta'}_{\alpha'} + j_{\alpha'}j^{\beta'})\,G_{a\beta\beta'}(x-x'), \\ &[(x\,|\,{}_{\alpha}\varphi'_{\beta tr}\,|\,y_0\},\{y_0\,|\,{}^{\beta}\varphi'^{\dagger}_{\alpha tr}\,|\,x')]^{-} = \\ &= (g_{\alpha\alpha'}j^{\beta}j^{\beta'} - j_{\alpha}j^{\beta}\delta^{\beta'}_{\alpha'} - j_{\alpha'}j^{\beta'}\delta^{\beta}_{\alpha} - \delta^{\beta}_{\alpha}\delta^{\beta'}_{\alpha'})\,G_{a\beta\beta'}(x-x'), \end{aligned}\right\} \quad . \quad (44)$$

all other commutators give zero. $G_{\beta\beta'}(x)$ in (44) is together with $F(x)$

and $G(x)$, which will be used further on, defined by

$$\left.\begin{aligned} F_{\underset{a}{s}}(x) &= \frac{i}{\hbar c}\,\frac{1}{j_\gamma\,\partial/\partial x_\gamma}\,D_{\underset{s}{a}}(x), \\ G_{\underset{s}{a}}(x) &= \left(\frac{i}{\hbar c}\right)^2 \frac{1}{(j_\gamma\,\partial/\partial x_\gamma)^2}\,D_{\underset{s}{a}}(x), \\ G_{\underset{s}{a}\beta\beta'}(x) &= \left(\frac{\hbar c}{i}\right)^2 \partial^2/\partial x_\beta\,\partial x_{\beta'}\,G_{\underset{s}{a}}(x). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45)$$

For $F_s(x)$ we get outside the lightcone

$$F_s(x) = \left(\frac{i}{\hbar c}\right)^2 \frac{1}{4\pi}\,\frac{1}{(x_\alpha x^\alpha - (j_\beta x^\beta)^2)^{\frac{1}{2}}} \quad (\text{for } x > |t|) \quad . \quad . \quad (46)$$

and zero inside (for $x < |t|$). The $G_{\beta\beta'}$ satisfy the relations

$$\left.\begin{aligned} j^\beta j^{\beta'}\,G_{\underset{s}{a}\beta\beta'}(x) &= D_{\underset{s}{a}}(x), \\ g^{\beta\beta'}\,G_{\underset{s}{a}\beta\beta'}(x) &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The function $W_{\alpha\alpha'}$ in (18) (which in the unitary description becomes the interaction function) splits up into

$$W_{\alpha\alpha'} = W_{\alpha\alpha' sc} + W_{\alpha\alpha' ln} + W_{\alpha\alpha' tr}, \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where

$$\left.\begin{array}{l|l|l|l} & \dfrac{i}{hc}\,W_{\alpha\alpha' sc} & \dfrac{i}{hc}\,W_{\alpha\alpha' ln} & \dfrac{i}{hc}\,W_{\alpha\alpha' tr} \\ \hline Dd & -j_\alpha j^\beta\,j_{\alpha'} j^{\beta'} G^\sigma_{a\beta\beta'} & \multirow{2}{*}{$(j_\alpha j^\beta + \delta^\beta_\alpha)(j_{\alpha'} j^{\beta'} + \delta^{\beta'}_{\alpha'})\,G^\sigma_{a\beta\beta'}$} & \multirow{2}{*}{$(g_{\alpha\alpha'} j^\beta j^{\beta'} - \delta^\beta_\alpha \delta^{\beta'}_{\alpha'} - \delta^\beta_\alpha j_{\alpha'} j^{\beta'} - j_\alpha j^\beta \delta^{\beta'}_{\alpha'})\,G^\sigma_{a\beta\beta'}$} \\ Dh & +j_\alpha j^\beta\,j_{\alpha'} j^{\beta'} G^\sigma_{a\beta\beta'} & & \\ \hline Hd & -j_\alpha j^\beta\,j_{\alpha'} j^{\beta'} G^\sigma_{+\beta\beta'} & \multirow{2}{*}{$(j_\alpha j^\beta + \delta^\beta_\alpha)(j_{\alpha'} j^{\beta'} + \delta^{\beta'}_{\alpha'})\,G^\sigma_{+\beta\beta'}$} & \multirow{2}{*}{$(g_{\alpha\alpha'} j^\beta j^{\beta'} - \delta^\beta_\alpha \delta^{\beta'}_{\alpha'} - \delta^\beta_\alpha j_{\alpha'} j^{\beta'} - j_\alpha j^\beta \delta^{\beta'}_{\alpha'})\,G^\sigma_{+\beta\beta'}$} \\ Hh & +j_\alpha j^\beta\,j_{\alpha'} j^{\beta'} G^\sigma_{-\beta\beta} & & \end{array}\right\} \quad (49)$$

the arguments being the same as in (22).

4.12 *Supplementary conditions.* If the 4-vector wave function ${}^\alpha\varphi$ is a solution of (4), the 4-scalar wave function

$$\frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\,{}^\alpha\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

is also a solution. The states (50) are kept out of the Ψ's by the supplementary conditions.

The two operators

$$\left.\begin{aligned} \mathbf{C}(x_1, \ldots x_n; x) &= \frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\,\Phi^\alpha(x) + e\,2\pi(\hbar c)^2 \sum_{l=1}^{n} 2F_s(x - x_l), \\ \mathbf{C}_{rd}(x_1, \ldots x_n; x) &= \frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\,\Phi^\alpha_{rd}(x) \end{aligned}\right\} \quad (51)$$

commute in all points with each other and with $\mathbf{K}_k^0\{x_k\} + \mathbf{K}^1(x_k)$. Therefore they are for arbitrarily fixed x^α integrals of electron motion at least as long as all electron world points lie outside each others light cone. So we can write

$$\left.\begin{aligned}\mathbf{C}(x_1,\ldots x_n;x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi &= \mathbf{C}(x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi,\\ \mathbf{C}_{rd}(x_1,\ldots x_n;x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi &= \mathbf{C}_{rd}(x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi.\end{aligned}\right\} \quad (52)$$

Because $(\partial^2/\partial x^\alpha\,\partial x_\alpha)\,\mathbf{C} = (\partial^2/\partial x^\alpha\,\partial x_\alpha)\,\mathbf{C}_{rd} = 0$ the conditions

$$\left.\begin{aligned}\mathbf{C}(x_1,\ldots x_n;x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi &= 0,\\ \mathbf{C}_{rd}(x_1,\ldots x_n;x)\,(x_1,\ldots x_n;y_{10},\ldots|\,\Psi &= 0\end{aligned}\right\} \quad \ldots \quad (53)$$

are satisfied for every t if at an arbitrarily given t they are satisfied together with their first derivative in t. They are a generalization of the elementary conditions (3) for the charge free case.

4.13 *Elimination of non-transverse wave operators.* The non-transverse wave operators can be eliminated from the equations of motion in various ways [11) 13)]. The procedure consists of two steps. One of these steps makes use of the supplementary conditions, the other step can be made by means of a gauge transformation. In behalf of later comparison with the unitary description we shall first make the latter step.

4.131 *Transformed equations of motion.* For our present purpose a suitable transformation is

$$(x_1,\ldots x_n;y_{10},\ldots|\,\Psi = e^{-\frac{ie}{\hbar c}\sum_{l=1}^{n}\frac{j_\alpha}{j_\gamma\,\partial/\partial x_\gamma}\Phi^\alpha(x_l)}(x_1,\ldots x_n;y_{10},\ldots|\,\Psi^*. \quad (54)$$

By this transformation an operator $\mathbf{Q}$ is transformed into

$$\mathbf{Q}^* = e^{\frac{ie}{\hbar c}\sum_{l=1}^{n}\frac{j_\alpha}{j_\gamma\,\partial/\partial x_\gamma}\Phi^\alpha(x_l)}\,\mathbf{Q}\,e^{-\frac{ie}{\hbar c}\sum_{l=1}^{n}\frac{j_\alpha}{j_\gamma\,\partial/\partial x_\gamma}\Phi^\alpha(x_l)} \quad \ldots \quad (55)$$

If we apply this to the operators $\mathbf{C}$ and $\mathbf{C}_{rd}$ of (51) we get with the help of the commutation relations (14)

$$\left.\begin{aligned}\mathbf{C}^*(x_1,\ldots x_n;x) &= \frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\,\Phi^\alpha(x),\\ \mathbf{C}^*_{rd}(x_1,\ldots x_n;x) &= \frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\,\Phi^\alpha_{rd}(x).\end{aligned}\right\} \quad \ldots \quad (51^*)$$

In the same way we obtain

$$\left(\frac{\hbar c}{i}\frac{\partial}{\partial x^\alpha} + e\,\Phi_\alpha(x)\right)^* = \frac{\hbar c}{i}\frac{\partial}{\partial x_\alpha} + e\,\Phi^{(*)}_\alpha(x), \quad \ldots \quad (56^*)$$

where $\Phi^{(*)}_\alpha$ stands for

$$\left.\begin{aligned}\Phi^{(*)}_\alpha(x) = \Phi_{\alpha tr}(x) + \left(j_\alpha + \frac{\partial/\partial x^\alpha}{j_\gamma\,\partial/\partial x_\gamma}\right)\frac{\partial/\partial x_\beta}{j_\delta\,\partial/\partial x_\delta}\,\Phi_\beta(x) -\\ -e^2\,2\pi(\hbar c)^2\,j_\alpha\sum_{l=1}^{n}2F_s(x-x_l),\end{aligned}\right\} \quad . \quad (57^*)$$

which is not the transformed Φ^*_α of Φ_α alone.

The principle of gauge invariance requires that the 4-gradient term in (57*) could also be removed. This can be done by the further transformation

$$(x_1,\ldots x_n;y_{10},\ldots\,|\,\Psi^*=e^{-\frac{ie}{\hbar c}\sum_{l=1}^{n}\frac{\partial/\partial x^\alpha}{(j_\gamma\,\partial/\partial x_\gamma)^2}\Phi^\alpha(x_l)}(x_1,\ldots x_n;y_{10},\ldots\,|\,\Psi^{**}. \qquad (54^*)$$

The operators (51*) remain invariant under this transformation

$$\mathbf{C}^{**}=\mathbf{C}^* \quad , \quad \mathbf{C}^{**}_{rd}=\mathbf{C}^*_{rd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51^{**})$$

and (56*) transforms into (56**) with two asteriks instead of one and $\Phi_\alpha^{(**)}$ standing for

$$\Phi_\alpha^{(**)}(x)=\Phi_{\alpha tr}(x)+j_\alpha\frac{\partial/\partial x_\beta}{j_\gamma\,\partial/\partial x_\gamma}\Phi_\beta(x)-e^2\,2\pi(\hbar c)^2\,j_\alpha\sum_{l=1}^{n}2F_s(x-x_l). \qquad (57^{**})$$

The transformed equations of motion (36**) for Ψ^{**} can be obtained from the original equations (36) for Ψ in replacing Φ_α by $\Phi_\alpha^{(**)}$.

4.132 *Effective equations of motion.* Operating on transformed wave functions Ψ^* or Ψ^{**} which satisfy the transformed supplementary conditions (53*) or (53**), the operator $\Phi_\alpha^{(*)}$ (57*) or $\Phi_\alpha^{(**)}$ (57**) becomes effectively equal to

$$\Phi^{(*)}_{\alpha ef}(x)=\Phi_{\alpha tr}(x)-e^2\,2\pi(\hbar c)^2\,j_\alpha\sum_{l=1}^{n}2F_s(x-x_l). \quad . \quad . \quad (57^*_{ef})$$

The effective equations of motion (36^*_{ef}) for Ψ^* (or Ψ^{**} which is effectively the same) are obtained from the original equations (36) for Ψ in replacing Φ_α by $\Phi^{(*)}_{\alpha ef}$.

4.14 COULOMB *interaction.* By the two successive steps made in 4.131 and 4.132 the non-transverse wave operators have entirely been eliminated fromt he equations of motion. The effect of the non-transverse photon states is represented by the second term in the right hand member of (57^*_{ef}). According to (46) this term is equal to

$$e^2\,j_\alpha\sum_{l=1}^{n}\frac{1}{((x_\beta-x_{l\beta})\,(x^\beta-x_l^\beta)-(j_\gamma(x^\gamma-x_l^\gamma)^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (58)$$

where the sum is only extended over those points x_l^β, which lie outside the light cone of x^β ($|\,x-x_l\,|>c\,|\,t-t_l\,|$).

Practical applications of the dualistic theory are commonly in single-time description. If the effective equations of motion are reduced to single-time description with the time axis in the direction of j^α, the term (58) is reduced to

$$e^2\,\delta^0_\alpha\sum_{l=1}^{n}\frac{1}{|\vec{x}-\vec{x}_l|} \quad (t_l=t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and represents simultaneous COULOMB interaction.

The transverse part of (57^*_{ef}) leads to the proper radiation processes.

4.2 *Unitary theory.* There are two reasons why it would not be correct

to infer transformed and effective unitary equations of motion from the transformed and effective dualistic equations of motion in the same way as the original unitary equations (39) have been infered from the original dualistic equations (36). Neither the empty non-transverse states, nor the order in time of the non-transverse wave operators are invariant under the transformations. Therefore the unitary derivation has to start from the original unitary equations (39) straightaway. Meanwhile it is worth while to keep an eye on the dualistic derivation during the course of the unitary one.

4.21 *Reduced equations of motion.* In the equation of motion (39) we split up $W_{\alpha\alpha'}$ according to (49) into a transverse part $W_{\alpha\alpha' tr}$ and a non-transverse part $W_{\alpha\alpha' nt} = W_{\alpha\alpha' sc} + W_{\alpha\alpha' ln}$. $W_{\alpha\alpha' tr}$ will be left unchanged throughout the reduction. In $W_{\alpha\alpha' nt}$ we insert

$$\left.\begin{aligned} \delta_\alpha^{\bar{\beta}}\,\delta_{\alpha'}^{\bar{\beta}'}\,G^\sigma_{a\bar{\beta}\bar{\beta}'}(x) &= \left(\frac{\hbar c}{i}\right)^2 \frac{\partial^2}{\partial x^\alpha \partial x^{\alpha'}}\,G^\sigma_a(x) - i_\alpha j_{\alpha'}\,2\,\delta(j_\gamma x^\gamma)\,\frac{\hbar c}{i}\,F^\sigma_s(x), \\ \delta_\alpha^{\beta}\,\delta_{\alpha'}^{\beta'}\,G^\sigma_{s'\beta\beta'}(x) &= \left(\frac{\hbar c}{i}\right)^2 \frac{\partial^2}{\partial x^\alpha\,\partial x^{\alpha'}}\,G^\sigma_s(x) \end{aligned}\right\} \quad (60)$$

and

$$\left.\begin{aligned} j_\alpha j^\beta\,\delta_{\alpha'}^{\beta'}\,G^\sigma_{a\beta\beta'}(x) &= j_\alpha\,\frac{\hbar c}{i}\,\frac{\partial}{\partial x^{\alpha'}}\,\sigma(x)\,F^\sigma_s(x) + j_\alpha j_{\alpha'}\,\frac{\hbar c}{i}\,2\,\delta(j_\gamma x^\gamma)\,\frac{\hbar c}{i}\,F^\sigma_s(x), \\ j_\alpha j^\beta\,\delta_{\alpha'}^{\beta'}\,G^\sigma_{s\beta\beta'}(x) &= j_\alpha\,\frac{\hbar c}{i}\,\frac{\partial}{\partial x^{\alpha'}}\,\sigma(x)\,F^\sigma_a(x). \end{aligned}\right\} \quad (61)$$

In the exponent in (39) the sum

$$\left.\begin{aligned} &\frac{ie^2}{\hbar c 2}\sum_{k,l=1}^{n}\sum_{m=1}^{p}\left(x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha}\right)\left(x_l^{(m_l+1)\alpha'} - x_l^{(m_l)\alpha'}\right) \\ &\qquad \left(\frac{\hbar c}{i}\right)^2 \frac{\partial^2}{\partial x^\alpha_{kl}\,\partial x^{\alpha'}_{kl}}\,G^\sigma\left(x_k^{(m'_k)} - x_l^{(m'_l)}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (62)$$

where x_{kl} stands for $x_k - x_l$, might in the limit

$$p_k \to \infty,\ \max\left|t_k^{(m_k+1)} - t_k^{(m_k)}\right| \to 0$$

be treated as

$$-\frac{ie^2}{\hbar c 2}\sum_{k,l=1}^{n}\int_{(x')}^{(x)}\!\!\int dx''_k\,dx''_l\left(\frac{\hbar c}{i}\right)^2 \frac{\partial^2}{\partial x_k^\alpha\,\partial x_l^{\alpha'}}\,G^\sigma(x''_k - x''_l). \quad . \quad . \quad . \quad (63)$$

It could already for finite initial and final times t'_k and t_k be removed by the transformation

$$(x_1, \ldots x_n|\,\Psi = e^{\frac{ie^2\hbar c}{2}\sum_{k,l=1}^{n} G^\sigma(x_k - x_l)}\,(x_1, \ldots x_n|\,\Psi^* \quad . \quad . \quad . \quad (64)$$

In (39), where for all electrons but one (the rth one) the initial and final times are shifted to $-\infty$ and $+\infty$ respectively, the contributions of (62) may be assumed to vanish even without this transformation. The same can

be assumed for the contributions of the terms

$$\left.\begin{aligned}&\frac{ie^2}{\hbar c 2}\sum_{k,l=1}^{n}\sum_{m=1}^{p}(x_k^{(m+1)\alpha}-x_k^{(m_k)\alpha})\,(x_l^{(m_l+1)\alpha'}-x_l^{(m)\alpha'})\\&\left(j_\alpha\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha'}}+j_{\alpha'}\frac{\hbar c}{i}\frac{\partial}{\partial x_k^{\alpha}}\right)\sigma(x_k^{(m'_k)}-x_l^{(m'_l)})\,F^\sigma(x_k^{(m'_k)}-x_l^{(m'_l)}),\end{aligned}\right\}\quad(65)$$

except for those in which the differentiation is with respect to x_r. The latter ones might be replaced by

$$\left.\begin{aligned}&\frac{e^2}{2}\sum_{l=1}^{n}\sum_{m=1}^{p}(x_l^{(m_l+1)\alpha'}-x_l^{(m_l)\alpha'})\,j_{\alpha'}\,(\sigma(x_r-x_l^{(m'_l)})\,F^\sigma(x_r-x_l^{(m'_l)})\\&\qquad-\sigma(x'_r-x_l^{(m'_l)})\,F^\sigma(x'_r-x_l^{(m'_l)})).\end{aligned}\right\}\quad\ldots\quad(66)$$

Once more because in (39) the initial and final times of all electrons except the rth one are shifted to $-\infty$ and $+\infty$ the contribution of the terms (66) can be removed by the transformation

$$(x_1,\ldots x_n|\,\Psi=e^{-\frac{e^2}{2}\sum_{k,l=1}^{n}\int_{(x'_l)}^{(x_l)}j_\alpha\,dx_l''^{\alpha}\,\sigma(x_k-x_l'')\,F^\sigma(x_k-x_l'')}(x_1,\ldots x_n|\,\Psi^*.\quad(67)$$

By that all 4-gradiant terms of $W_{\alpha\alpha' nt}$ have been removed. The remaining terms give the reduced non-transverse interaction function $W^*_{\alpha\alpha' nt}$

$$\frac{i}{\hbar c}\,W^*_{\alpha\alpha' nt}(x,x')=j_\alpha j_{\alpha'}(2\delta j_\gamma(x^\gamma-x'^\gamma))\frac{\hbar c}{i}F_s(x-x')+B(x-x'))\quad(68)$$

with

	$B(x)$
Dd	0
Dh	$D_a^\sigma(x)$
Hd	0
Hh	$D_s^\sigma(x).$

$\ldots\ldots\ldots\ldots\quad(69)$

The reduced equations of motion (39*) are obtained from the original ones (39) in replacing $W_{\alpha\alpha'}$ by $W_{\alpha\alpha' tr}+W^*_{\alpha\alpha' nt}$.

4.22 COULOMB *interaction*. After the reduction the non-transverse interaction is represented by (68). The part which the term with F_s contributes to the exponent of (39*) is according to (46)

$$\left.\begin{aligned}&\frac{ie^2}{\hbar c 2}\sum_{k,l=1}^{n}\sum_{m=1}^{p}j_\alpha(x_k^{(m_k+1)\alpha}-x_k^{(m_k)\alpha})\,j_{\alpha'}(x_l^{(m_l+1)\alpha'}-x_l^{(m_l)\alpha'})\,\delta(j_\gamma(x_k^{(m'_k)\gamma}-x_l^{(m'_l)\gamma})\\&\qquad\frac{1}{((x_k^{(m'_k)\delta}-x_l^{(m'_l)\delta})\,(x_{k\delta}^{(m'_k)}-x_{l\delta}^{(m'_l)}))^{\frac{1}{2}}}.\end{aligned}\right\}\quad(70)$$

If the time axis is chosen in the direction of j^α (70) reads

$$\frac{ie^2}{\hbar 2}\sum_{k,l=1}^{n}\sum_{m=1}^{p}(t_k^{(m_k+1)}-t_k^{(m_k)})\,(t_l^{(m_l+1)}-t_l^{(m_l)})\,\delta(t_k^{(m'_k)}-t_l^{(m'_l)})\frac{1}{|\vec{x}_k^{(m'_k)}-\vec{x}_l^{(m'_l)}|}.\quad(71)$$

Even in many-times description this represents simultaneous COULOMB interaction.

In case *Dh* the term with D_a^σ contributes according to (11) a part, which represents a similar COULOMB interaction along the past and future branches of the light cone. In case *Hh* the term with D_s^σ leads according to (11) to a different type of interaction, which extends over all time-space with a singularity along the light cone.

The transverse interaction for *Dd* is the same as for *Dh*; for *Hd* it is the same as for *Hh*. Though we now don't talk of radiation, we might call it radiative interaction.

4.3 *Comparison of dualistic and unitary reduction.* 4.31 *Transformed equations.* If from the transformed dualistic equations of motion (36**) we would thoughtlessly try to infer unitary equations in the same way as the equations (39) have been infered from (36), the first two terms of $\Phi_\alpha^{(**)}$ (57**) would just yield the interaction function $W_{\alpha\alpha' tr} + W^*_{\alpha\alpha' nt}$, which is the interaction function in the reduced unitary equations of motion (39*). The latter, however, does not contain a further part, which could be considered as a counterpart of the remaining terms with F_s in (57**). In fact it has been argued in 4.2 that this thoughtless procedure cannot be expected to be correct.

4.32 *Effective equations.* The objections against the application of the same procedure to the effective dualistic equations of motion (36^*_{ef}) are less stringent because in the latter the non-transverse wave operators have entirely been removed and the transverse ones have been left unaffected. The first term of $\Phi^{(*)}_{\alpha ef}$ (57^*_{ef}) would yield the radiative interaction function $W_{\alpha\alpha' tr}$. Now the remaining part $W^*_{\alpha\alpha' nt}$ of the interaction function in (39*) can in the cases *Dd* and *Hd*, in which it represents instantaneous COULOMB interaction, more or less be considered as the counterpart of the remaining terms with F_s in (57^*_{ef}), which in those cases in which single-time description is used also represent instantaneous COULOMB interaction.

Therefore if anything could be expected of a unitary description, the cases *Dd* or *Hd* might be hoped to give the same results as the dualistic description, whereas in the cases *Dh* or *Hh* one might expect additional effects represented by D_a^σ or D_s^σ. Just the cases *Dh* and *Hh* have been rejected in 3.35 because they depend on j^a already before the distinction into transverse and non-transverse interaction has been made.

4.33 *Supplementary conditions.* It should be noticed that, whereas in the dualistic theory supplementary conditions were needed in order to reduce the total field to the transverse field and the COULOMB field, in the unitary theory the reduction of the total interaction to the transverse interaction and the COULOMB interaction has been made without supplementary conditions. (Also in classical theory the 4-divergence of the field potential vanishes without extra conditions if the field is derived from a unitary charge description).

5. **Transitions.**

5.1 *Transition representation.* Up till now we have used a time-space representation based on FEYNMAN's calculus. Everywhere in the exponents of (37) as well as (39) and (39*) $(x_k^{(m_k+1)\alpha} - x_k^{(m_k)\alpha})$ can be replaced by $\alpha_k^\alpha c\,(t_k^{(m_k+1)} - t_k^{(m_k)})$, provided the **N**'s and the exponentials are well ordered in time together instead of separately because they no longer commute. The latter restriction can easily be satisfied in (37) but in (39) and (39*) it requires that the exponentials shall be placed straddle-legged between the **N**'s in a way, which cannot simply be symbolized in a formula. It is just a peculiar feature of FEYNMAN's calculus that thanks to the properties of **N** one can instead of the $\alpha^\alpha ct$ exponents use the x^α exponents, which do commute with the **N**'s.

Now another way to get rid of the difficulties with the ordering of the **N**'s and the exponentials is to get rid of the **N**'s (29) by means of a unitary transformation. The transformation nucleus is supplied by a complete orthonormal system of solutions of the unperturbed electron equations (34) and in the dualistic theory also by complete systems of free particle wave functions (5). The transformed representation expresses in the unitary theory the interaction between the electrons in terms of transitions between unperturbed electron states and in the dualistic theory it expresses the interaction between electrons and photons in terms of transitions between unperturbed electron states together with creation and annihilation of free photons. Therefore this representation will be called the transition representation. If the interaction is small compared with the unperturbed electron motion, the transition representation provides a direct base for perturbation calculus.

We shall write down the transition representation only for the unitary description.

5.2 *Transformation.* In the transition representation the wave functions $(x_1, \ldots x_n \mid \Psi$ are represented by their expansion coefficients with respect to the products $(x_1 \mid \psi_1^0 \mid \mu_1) \ldots (x_n \mid \psi_n^0 \mid \mu_n)$ of unperturbed individual wave functions. These coefficients are given by

$$(\mu_1 t_1, \ldots \mu_n t_n \mid \Psi =$$

$$= \int \ldots \int (d\vec{x}_1) \ldots (d\vec{x}_n)\, (\mu_n \mid \psi_n^{\dagger 0} \mid x_n) \ldots (\mu_1 \mid \psi_1^{\dagger 0} \mid x_1)\, (x_1, \ldots x_n \mid \Psi. \quad (72)$$

A 2-electron operator $\mathbf{Q}\{x_k, x_l\}$ is represented by $\mathbf{Q}\{\mu_k t_k, \mu_l t_l\}$, which is given by

$$\mathbf{Q}\{\mu_k t_k, \mu_l t_l\}(\mu_1 t_1, \ldots \mu_n t_n \mid \Psi = \sum_{\mu_k', \mu_l'} (\mu_k t_k, \mu_l t_l \mid \mathbf{Q}\{x_k, x_l\} \mid \mu_k' t_k, \mu_l' t_l)$$

$$(\mu_l t_l, \ldots \mu_k' t_k, \ldots \mu_l' t_l, \ldots \mu_n t_n \mid \Psi, \quad . \quad . \quad . \quad (73)$$

with the matrix elements

$$(\mu_k t_k, \mu_l t_l \mid \mathbf{Q}\{x_k, x_l\} \mid \mu_k' t_k, \mu_l' t_l) =$$

$$= \int\int (d\vec{x}_k)(d\vec{x}_l)(\mu_l \mid \psi_l^\dagger \mid x_l)(\mu_k \mid \psi_k^\dagger \mid x_k)\mathbf{Q}\{x_k, x_l\}(x_k \mid \psi_k \mid \mu_k')(x_l \mid \psi_l \mid \mu_l'). \quad (74)$$

The generalization to 1- and more-electron operators is obvious.

5.3 *Transition equations.* Because in the equations of motion (39) and (39*) the N's are removed by the transformation, the limiting process can now formally be performed by replacing the sums in the exponents by integrals. The latter have still to be understood symbolically as the integrants are well ordered in time. Just as the sums of exponents actually stand for products of exponentials, the integrals stand for productals [14]). The productants are well ordered in time.

The original equations of motion (39) are transformed into the transition equations

$$\left.\begin{array}{l} (\mu_1 t_1, \ldots \mu_n t_n \mid \Psi_{t_1=\ldots t_{r-1}=t_{r+1}=\ldots t_n \to +\infty} \\ e^{\frac{ie^2}{\hbar c}\frac{1}{2}\int_{t'_k}^{t_k}\int_{t'_l}^{t_l} c\,dt''_k c\,dt''_l\, \alpha_k^{\alpha}(\mu_k t''_k \mid \alpha_l^{\alpha'}(\mu_l t''_l \mid W_{\alpha\alpha'}(\mu_k t''_k, \mu_l t''_l)} \\ (\mu_1 t'_1, \ldots \mu_n t'_n \mid \Psi_{t_1=\ldots t_{r-1}=t_{r+1}=\ldots t_n \to -\infty} \qquad (r=1, \ldots n). \end{array}\right\} . \quad (75)$$

The reduced transition equations (75*) are obtained from the original ones (75) in replacing $W_{\alpha\alpha'}$ by $W_{\alpha\alpha' tr} + W^*_{\alpha\alpha' nt}$.

The transition representation makes no use of FEYNMAN's imperfect calculus. It is particularly suitable for applications such as in 3. The time-space representation, which has been made possible by FEYNMAN's calculus, is the more appropriate form for dealing with general topics such as the present derivations and problems of electron self interaction and pair processes, belonging to Q_3.

5.4 COULOMB *interaction.* As the COULOMB interaction represented in (75*) by $W^*_{\alpha\alpha' nt}$ is in general not small compared with the unperturbed electron motion, it is often desirable to include it in the unperturbed motion. The unperturbed motions of the various electrons are then no longer independent of each other and the wave function of the entire system can no longer be written as a product of individual wave functions of separate electrons. The separation into individual functions can in many cases be restored by an approximation in which for each electron the instantaneous COULOMB fields due to the other electrons are replaced by an appropriate average field. For different electrons we get in general different average fields. The approximation will fail as soon as the correlations between the unperturbed motions of the various electrons begin to act a part. The average fields together with the fields due to other kinds of charges (e.g. protons) constitute the unperturbed fields A_k in (34).

If the radiative interaction represented by the remaining interaction function $W_{\alpha\alpha' tr}$ in (75*) is sufficiently small, it can be treated as a perturbation.

5.5 *Paired transitions.* Remembering (73) we see that the transition processes described by (75*) are composed of elementary paired transitions of ever 2 electrons during infinitesimal time intervals. These paired transitions and their compositions will be discussed in 3.

6. Conclusion.

6.1 *D-modification or H-revision.* Arguments against the cases *Dh* and *Hh* have been given in 3.35 and 4.32. A further argument can be found from a comparison between the photon case and the case of neutral carrier particles with spin 1 and $m \neq 0$ (neutral mesons) of which the photon case is a degenerate case ($m = 0$). For $m \neq 0$ there occur no negative vectors, only negative energy states. In *D*-modification and in *H*-revision the interaction function $W_{\alpha\alpha'}$ is given by

$$\left.\begin{array}{c|c} & \frac{i}{\hbar c} W_{\alpha\alpha'}(x, x') \\ \hline D & g_{\alpha\alpha'} D_a^{\sigma}(x-x') \\ H & g_{\alpha\alpha'} D_+^{\sigma}(x-x'), \end{array}\right\} \quad \ldots\ldots\ldots \quad (76)$$

with D_a and D_s given by (1 27). For $m \to 0$ the interaction functions for the cases *D* and *H* degenerate into the photon interaction functions respectively for the cases *Dd* and *Hd*. Therefore all evidence is against the cases *Dh* and *Hh*.

It has been mentioned already in 3.36 that it will be seen in 3 that the cases *Dd* and *Dh* are excluded by the asymmetry condition of 1.51.

All things considered, the only case which might leave some hope for unification is *Hd*. That means that for photons the negative energy states have to be treated according to hole theory (*H*-revision), the negative vectors according to the original theory (*D*-modification).

6.2 *Equations of motion.* If a unitary theory were feasible at all, it might be expected to be characterized by integral equations of motion of the type (39). This is an intractable type. The main difficulties lie in the consistency and boundary (initial) conditions.

6.3 *Introduction of photons.* If the unitary theory had been completely established, fictitious photon states with wave operators to match could be introduced as in 2.1. The exponential function in (39) could be written as the empty-empty part of the exponential operator in (37). Because no photons are supposed to be present in the initial and final states we could take the entire operator as well. Doing so we could then cut off at finite initial and final times not only for 1 but for all electrons at a time. If all this were allowed, we would have obtained a dualistic description. The correct form of the empty-empty part could only be obtained with BOSE–EINSTEIN statistics. The fundamental part of these statistics will become still more conspicuous in the treatment of transition processes in 3.

In the dualized description (52) would hold but (53) in general would not. Now in the unitary description the interaction function could be reduced to radiative interaction and instantaneous COULOMB interaction. In the dualized description a corresponding reduction would require the introduction of the supplementary conditions (53).

6.4 *Applications.* The solution of the consistency and boundary problems has to decide about the applicability of the unitary description. The results of the practical applications have to decide about the correctness of the unitary description as compared with the dualistic one.

6.5 *Generalization.* The present tentative treatment of the photon case could in principle be generalized to other kinds of carrier particles, provided they obey BOSE–EINSTEIN statistics. For neutral mesons no essentially new difficulties occur. For charged mesons, however, there is a further complication. During the time between their emission and absorption these particles cannot be treated as free because of the presence of other charges. That makes that in the unitary description the interaction function $W(x, x')$ will not only depend on the variables (x) and (x') of the interacting source particles (nucleons), but also on the electromagnetic field (or on all electric charges). That would considerably aggravate the intractability of the description.

Summary.

If a unitary charge theory with advanced and retarded interaction at a distance is possible at all, it will be almost intractable anyhow. In spite of this it has been asked how such a theory would have to look like. Even an imperfect answer might show some aspects of the dualistic charge-field theory, in particular about the properties of the field, which are less conspicuous in the common description. The present paper deals with the presumable type of the equations of motion of electrons. Further after the interaction has been split up into a transverse and a non-transverse part, the latter part has been reduced to simultaneous COULOMB interaction.

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Mathematics. — *A non-homogeneous inequality for integers in a special cubic field.* (First communication.) By A. V. PRASAD. (Communicated by Prof. J. G. VAN DER CORPUT.)

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1. Let ξ, η, ζ be three real linear forms in integral variables u, v, w with determinant $\Delta \neq 0$. Let $\mathcal{M}(\xi, \eta, \zeta)$ be the lower bound of the numbers λ such that, if a, b, c are any real numbers, then there exist integers u, v, w satisfying

$$|(\xi - a)(\eta - b)(\zeta - c)| \leqslant \lambda |\Delta|.$$

Then it follows by a well known theorem of REMAK [1]) that

$$\mathcal{M}(\xi, \eta, \zeta) \leqslant \tfrac{1}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The value of $\mathcal{M}(\xi, \eta, \zeta)$ will in fact be $\frac{1}{8}$, when ξ, η, ζ have certain special forms; in particular when $\xi = u$, $\eta = v$, $\zeta = w$. DAVENPORT [2]) has proved that, if none of the forms ξ, η, ζ represents zero for integral values of u, v, w other than 0, 0, 0 then

$$\mathcal{M}(\xi, \eta, \zeta) < \tfrac{1}{8}.$$

Further DAVENPORT [3]) determines the exact value of $\mathcal{M}(\xi, \eta, \zeta)$ in the two cases when

$$\xi = u + \theta_r v + \theta_r^2 w, \eta = u + \phi_r v + \phi_r^2 w, \zeta = u + \psi_r v + \psi_r^2 w, \quad (2)$$

where θ_1, ϕ_1, ψ_1 are the roots of

$$t^3 + t^2 - 2t - 1 = 0,$$

and θ_2, ϕ_2, ψ_2 are the roots of

$$t^3 - 3t - 1 = 0.$$

In the first case the modulus of the determinant of the forms is 7 and $\mathcal{M}(\xi, \eta, \zeta) = \frac{1}{4\frac{1}{7}}$ and in the second case the modulus of the determinant is 9 and $\mathcal{M}(\xi, \eta, \zeta) = \frac{1}{2\frac{1}{7}}$. In each case the product of the three forms is the norm of the general integer of the cubic field $k(\theta_r)$; the fields $k(\theta_1)$ and $k(\theta_2)$ being the totally real cubic fields of least discriminants.

In this paper we are concerned with the product of one real linear form ξ and two conjugate complex linear forms $\eta, \bar{\eta}$ in integral variables u, v, w, with determinant $\Delta \neq 0$. Let $\mathcal{M}(\xi, \eta, \bar{\eta})$ be the lower bound of the numbers

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[1]) R. REMAK, Math. Zeitschrift, **17**, 1—34 (1923) and **18**, 173—200 (1923). For a simpler proof see H. DAVENPORT, Jour. London Math. Soc., **14**, 47—51 (1939).

[2]) H. DAVENPORT, Proc. Camb. Philos. Soc., **43**, 137—152 (1947).

[3]) Ibid.

λ such that, if a is real and b and $\bar{b}$ are conjugate complex, then there are integers u, v, w satisfying

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<\lambda|\Delta|.$$

Now there is no theorem corresponding to that of REMAK giving an absolute bound for $\mathcal{M}(\xi, \eta, \bar{\eta})$. There are forms $\xi, \eta, \bar{\eta}$ for which $\mathcal{M}(\xi, \eta, \bar{\eta})$ is arbitrarily large. In particular, if we take $\varepsilon>0$ and

$$\left.\begin{array}{lll} \xi=u, & \eta=v+i\varepsilon w, & \bar{\eta}=v-i\varepsilon w \\ a=\frac{1}{2}, & b=\frac{1}{2}, & \bar{b}=\frac{1}{2}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

then we have for all integers u, v, w

$$|(\xi-a)(\eta-b)(\bar{\eta}-b)|=|(u-\tfrac{1}{2})\{(v-\tfrac{1}{2})^2+\varepsilon^2 w^2\}|\geqq\tfrac{1}{8};$$

since the determinant of the forms (3) is $-2i\varepsilon$, this proves that

$$\mathcal{M}(\xi, \eta, \bar{\eta})\geqq\frac{1}{16\varepsilon}$$

for these forms. Hence $\mathcal{M}(\xi, \eta, \bar{\eta})$ can take arbitrarily large values. But it is still of interest to find the exact value of $\mathcal{M}(\xi, \eta, \bar{\eta})$ for particular forms $\xi, \eta, \bar{\eta}$. We consider the special forms

$$\xi=u+\theta v+\theta^2 w,\ \eta=u+\Phi v+\Phi^2 w,\ \bar{\eta}=u+\bar{\Phi}v+\bar{\Phi}^2 w, \quad . \quad (4)$$

where $\theta, \Phi, \bar{\Phi}$ are the real and complex roots of the equation

$$t^3-t-1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The determinant of these forms has modulus $\sqrt{(23)}$; we prove in this paper that

$$\mathcal{M}(\xi, \eta, \bar{\eta})=\frac{1}{5\sqrt{23}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These forms are of special interest as $\xi\eta\bar{\eta}$ is the norm of the general integer of the field $k(\theta)$, which is the cubic field with least discriminant.

Our result (6) can be regarded from a different point of view. We use ξ to denote any integer of the field $k(\theta)$ and we use η and $\bar{\eta}$ to denote the conjugates of ξ in the fields $k(\Phi)$ and $k(\bar{\Phi})$. Then since $1, \theta, \theta^2$ are a basis of the field $k(\theta)$, it follows that $\xi, \eta, \bar{\eta}$ can be expressed in the form (4) for suitable rational integers u, v, w. Conversely it is clear that, if $\xi, \eta, \bar{\eta}$ are given by (4), where u, v, w are any rational integers, then $\xi, \eta, \bar{\eta}$ are conjugate integers of the fields $k(\theta)$, $k(\Phi)$, $k(\bar{\Phi})$. Consequently $\mathcal{M}(\xi, \eta, \bar{\eta})\cdot|\Delta|$ is identical with the lower bound $\mathcal{M}(\theta)$ of the numbers λ such that for every real a and complex conjugate $b, \bar{b}$ there is an integer ξ of $k(\theta)$ such that

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<\lambda.$$

With this notation our result (6) becomes

$$\mathcal{M}(\theta)=\tfrac{1}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This result is amply sufficient to prove the existence of EUCLID's algorithm in the field $k(\theta)$. For, if α and β are any integers of $k(\theta)$ with $\beta \neq 0$, there is an integer ξ of $k(\theta)$ satisfying [4] $\left|N\left(\xi-\frac{\alpha}{\beta}\right)\right|<1$, and for this integer

$$|N(\alpha-\xi\beta)|<|N(\beta)|.$$

We are able in fact to prove a result rather more precise than (7). We prove the following theorem.

Theorem. *Let a be any real number and let b and $\bar{b}$ be any pair of conjugate complex numbers. Then there is an integer ξ of $k(\theta)$ with conjugates η and $\bar{\eta}$ in $k(\phi)$ and $k(\bar{\phi})$ for which*

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<\frac{1}{5\cdot 0001}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

unless a, b, $\bar{b}$ are of the form

$$a=\frac{\sigma}{\theta^2+1}+\xi_1, b=\frac{\tau}{\phi^2+1}+\eta_1, \bar{b}=\frac{\bar{\tau}}{\bar{\phi}^2+1}+\bar{\eta}_1, \quad . \quad . \quad . \quad (9)$$

where ξ_1, η_1, $\bar{\eta}_1$ are conjugate integers and σ, τ, $\bar{\tau}$ are conjugate units of the fields $k(\theta)$, $k(\phi)$, $k(\bar{\phi})$. If a, b, $\bar{b}$ are of the form (9) then there is no integer ξ of $k(\theta)$, with conjugates η, $\bar{\eta}$ satisfying

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<\tfrac{1}{5},$$

but there are an infinite number of integers ξ of $k(\theta)$, with conjugates η, $\bar{\eta}$ satisfying

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|=\tfrac{1}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The proof is based on the ideas used by DAVENPORT in the case of three real forms, but a new technique is needed to handle the inequalities in the present case, and to prove that the result is "isolated".

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2. *Outline of the proof of the theorem.*

Our proof depends on a number of lemmas. We give here the proof of the main theorem assuming the truth of the lemmas, which will be proved later.

Let a be any real number and let b and $\bar{b}$ be any pair of conjugate complex numbers.

Let $M=M(a,b,\bar{b})$ be the lower bound of

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|$$

[4] Here $N(\alpha)$ denotes as usual the norm of the number α of $k(\theta)$, i.e. the product of α by its conjugates in the fields $k(\phi)$ and $k(\bar{\phi})$.

for all integers ξ of $k(\theta)$. We suppose that

$$M \geqslant \frac{1}{5+\delta}, \quad \ldots \ldots \quad (11)$$

where we write, for convenience,

$$\delta = 0 \cdot 0001, \quad \ldots \ldots \quad (12)$$

and we deduce eventually that a, b, $\bar{b}$ must then be of the form (9). This will prove the first assertion of the theorem. From the definition of M it is clear that corresponding to any positive number ε_1, there is an integer ξ_1 of $k(\theta)$ satisfying

$$|(\xi_1 - a)(\eta_1 - b)(\bar{\eta}_1 - \bar{b})| = \frac{M}{1-\varepsilon}, \quad \ldots \ldots \quad (13)$$

where

$$0 \leqslant \varepsilon < \varepsilon_1. \quad \ldots \ldots \quad (14)$$

We suppose that ε_1 is sufficiently small; in the course of our proof we suppose that ε_1 is less than a finite number of positive absolute constants.

We write

$$\alpha = \frac{1}{a-\xi_1}, \quad \beta = \frac{1}{b-\eta_1}, \quad \bar{\beta} = \frac{1}{\bar{b}-\bar{\eta}_1}. \quad \ldots \ldots \quad (15)$$

Then, by (13) and (11),

$$|\alpha\beta\bar{\beta}| = \frac{1-\varepsilon}{M} \leqslant (5+\delta)(1-\varepsilon), \quad \ldots \ldots \quad (16)$$

and by (15), (16) and the definition of M,

$$\left.\begin{aligned} &|(\alpha\xi - 1)(\beta\eta - 1)(\bar{\beta}\bar{\eta} - 1)| \\ &\quad = |\alpha\beta\bar{\beta}| \cdot \left|\left(\xi - \frac{1}{\alpha}\right)\left(\eta - \frac{1}{\beta}\right)\left(\bar{\eta} - \frac{1}{\bar{\beta}}\right)\right| \\ &\quad = |\alpha\beta\bar{\beta}| \cdot |(\xi + \xi_1 - a)(\eta + \eta_1 - b)(\bar{\eta} + \bar{\eta}_1 - \bar{b})| \\ &\quad \geqslant (1-\varepsilon), \end{aligned}\right\} \quad \ldots \quad (17)$$

for all integers ξ of $k(\theta)$. Consequently we have

$$|(\alpha - \xi^{-1})(\beta - \eta^{-1})(\bar{\beta} - \bar{\eta}^{-1})| \geqslant (1-\varepsilon)\,|\xi\eta\bar{\eta}|^{-1}, \quad \ldots \quad (18)$$

for all integers $\xi \neq 0$ of $k(\theta)$.

Our object is to prove that, if α is real, and β, $\bar{\beta}$ are conjugate complex numbers, such that (16) is satisfied and (17) and (18) are satisfied for all integers $\xi \neq 0$ of $k(\theta)$ then α, β, $\bar{\beta}$ must be of certain special forms. It follows from Lemma 1 that there is no real loss of generality in assuming that

$$\frac{\alpha}{\theta^2} \leqslant |\beta|^2 \leqslant \alpha. \quad \ldots \ldots \quad (19)$$

Using this assumption together with (16) and (17) and (18) for various

integers $\xi \neq 0$ of $k(\theta)$, we prove by the Lemmas 2—20 that α, β, $\bar{\beta}$ are given by

$$\alpha = \theta^2 + 1,\ \beta = \phi^2 + 1,\ \bar{\beta} = \bar{\phi}^2 + 1 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

It follows by Lemma 1 that in the general case, when (19) is not necessarily satisfied, we must have

$$\alpha = \frac{(\theta^2+1)}{\sigma},\ \beta = \frac{(\phi^2+1)}{\tau},\ \bar{\beta} = \frac{(\bar{\phi}^2+1)}{\bar{\tau}}, \quad . \quad . \quad . \quad (21)$$

for some conjugate units σ, τ, $\bar{\tau}$ of the fields $k(\theta)$, $k(\phi)$, $k(\bar{\phi})$. It now follows by (15) that a, b, $\bar{b}$ are of the form (9). This completes the proof of the first assertion of the theorem.

The proof of the last assertion of the theorem is given as Lemma 21.

3. **Lemma 1.** *Let α be any real number and let β, $\bar{\beta}$ be any conjugate complex numbers such that $|\alpha\beta\bar{\beta}| \neq 0$ and*

(a) for every integer ξ of $k(\theta)$

$$|(\alpha\xi - 1)(\beta\eta - 1)(\bar{\beta}\bar{\eta} - 1)| \geqslant 1 - \varepsilon. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Then for some conjugate units $\sigma, \tau, \bar{\tau}$ of the fields $k(\theta)$, $k(\phi)$, $k(\bar{\phi})$ the numbers $\alpha^* = \alpha\sigma$, $\beta^* = \beta\tau$ and $\bar{\beta}^* = \bar{\beta}\bar{\tau}$ have the property (a) and satisfy

$$|\alpha^*\beta^*\bar{\beta}^*| = |\alpha\beta\bar{\beta}| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and

$$\frac{\alpha^*}{\theta^2} \leqslant |\beta^*|^2 < \alpha^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Proof. We obviously have $\alpha \neq 0$ and $\beta \neq 0$. We can choose a rational integer n so that

$$\frac{|\phi|^2}{\theta} \leqslant \frac{|\beta\phi^n|^2}{|\alpha\theta^n|} < 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We take

$$\sigma = \pm\theta^n,\ \tau = \pm\phi^n,\ \bar{\tau} = \pm\bar{\phi}^n. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

the sign being chosen to ensure that

$$\pm\alpha\theta^n > 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Then as θ, ϕ, $\bar{\phi}$ are conjugate units of $k(\theta)$, $k(\phi)$, $k(\bar{\phi})$ it follows that $\sigma, \tau, \bar{\tau}$ are conjugate units of these fields. It follows from (25), (26) and (27) that the numbers $\alpha^* = \alpha\sigma$, $\beta^* = \beta\tau$, $\bar{\beta}^* = \bar{\beta}\bar{\tau}$ satisfy (23) and (24). Further, if ξ is any integer of $k(\theta)$, $\sigma\xi$ is also an integer of $k(\theta)$ and so using (22) we have

$$|(\alpha^*\xi - 1)(\beta^*\eta - 1)(\bar{\beta}^*\bar{\eta} - 1)| = |(\alpha\sigma\xi - 1)(\beta\tau\eta - 1)(\bar{\beta}\bar{\tau}\bar{\eta} - 1)| \geqslant 1 - \varepsilon,$$

for all integers ξ of $k(\theta)$. Thus α^*, β^*, $\bar{\beta}^*$ have the property (a). This proves the lemma.

4. **Lemma 2.** *Let a be any real number and let b and $\bar{b}$ be any pair of conjugate complex numbers. Then there is an integer ξ of $k(\theta)$ satisfying*

$$|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})| \leqq \tfrac{8}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Proof. For any such a, b, $\bar{b}$ we define u_0, v_0, w_0 by the equations

$$\left.\begin{aligned} a &= u_0 + v_0\,\theta + w_0\,\theta^2, \\ b &= u_0 + v_0\,\Phi + w_0\,\Phi^2, \\ \bar{b} &= u_0 + v_0\,\bar{\Phi} + w_0\,\bar{\Phi}^2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

these equations having a unique solution, since the determinant

$$(\Phi-\bar{\Phi})(\bar{\Phi}-\theta)(\theta-\Phi)$$

of the forms on the right hand side does not vanish. Writing the equations (29) in the form

$$\left.\begin{aligned} a &= u_0 + v_0\,\theta + w_0\,\theta^2, \\ b+\bar{b} &= 2u_0 + v_0(\Phi+\bar{\Phi}) + w_0(\Phi^2+\bar{\Phi}^2), \\ i(b-\bar{b}) &= v_0\,i(\Phi-\bar{\Phi}) + w_0\,i(\Phi^2-\bar{\Phi}^2), \end{aligned}\right\}$$

we see that u_0, v_0, w_0 are real, b, $\bar{b}$ and Φ, $\bar{\Phi}$ being conjugate. We take

$$\xi = u + v\,\theta + w\,\theta^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where u, v, w are integers satisfying

$$|u-u_0| \leqq \tfrac{1}{2}, \quad |v-v_0| \leqq \tfrac{1}{2}, \quad |w-w_0| \leqq \tfrac{1}{2}.$$

Then ξ is an integer of $k(\theta)$ and writing

$$u-u_0 = x, \quad v-v_0 = y, \quad w-w_0 = z,$$

we have

$$\left.\begin{aligned} &(\xi-a)(\eta-b)(\bar{\eta}-\bar{b}) \\ &\quad = \{(u-u_0) + (v-v_0)\,\theta + (w-w_0)\,\theta^2\} \\ &\quad \times \{(u-u_0) + (v-v_0)\,\Phi + (w-w_0)\,\Phi^2\} \\ &\quad \times \{(u-u_0) + (v-v_0)\,\bar{\Phi} + (w-w_0)\,\bar{\Phi}^2\} \\ &\quad = (x + \theta y + \theta^2 z)(x + \Phi y + \Phi^2 z)(x + \bar{\Phi} y + \bar{\Phi}^2 z), \end{aligned}\right\} \quad . \quad (31)$$

where

$$|x| \leqq \tfrac{1}{2}, \quad |y| \leqq \tfrac{1}{2}, \quad |z| \leqq \tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Now multiplying out the expression on the right of (31) and using the fact that θ, Φ, $\bar{\Phi}$ are the roots of the equation (5) we obtain

$$(\xi-a)(\eta-b)(\bar{\eta}-\bar{b}) = x^3 + y^3 + z^3 + 2x^2 z - y^2 x + z^2 x - z^2 y - 3xyz. \quad (33)$$

We use (32) and (33) to show that the integer ξ of $k(\theta)$ given by (30) satisfies (28). In proving this we may clearly suppose without loss of generality that $x \geqq 0$. We consider four cases separately.

Case (i). When $x \geq 0$, $y \geq 0$, $z \geq 0$ we have

$$\begin{aligned} -\tfrac{5}{8} &\leqq -y^2x - z^2y - 3xyz \\ &\leqq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leqq x^3 + y^3 + z^3 + 2x^2z + z^2x \leqq \tfrac{6}{8}. \end{aligned}$$

Case (ii). When $x \geq 0$, $y \geq 0$, $z \leq 0$ we have

$$\begin{aligned} -\tfrac{5}{8} &\leqq -|z|^3 - 2x^2|z| - y^2x - z^2y \\ &\leqq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leqq x^3 + y^3 + z^2x + 3xy|z| \leqq \tfrac{6}{8}. \end{aligned}$$

Case (iii). When $x \geq 0$, $y \leq 0$, $z \geq 0$ we have

$$\begin{aligned} -\tfrac{2}{8} &\leqq -|y|^3 - y^2x \\ &\leqq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leqq x^3 + z^3 + 2x^2z + z^2x + z^2|y| + 3x|y|z \leqq \tfrac{9}{8}. \end{aligned}$$

Case (iv). When $x \geq 0$, $y \leq 0$, $z \leq 0$ we have

$$\begin{aligned} -\tfrac{8}{8} &\leqq -|y|^3 - |z|^3 - 2x^2|z| - y^2x - 3x|yz| \\ &\leqq (\xi - a)(\eta - b)(\bar{\eta} - \bar{b}) \\ &\leqq x^3 + z^2x + z^2|y| \leqq \tfrac{3}{8}. \end{aligned}$$

These results prove that ξ satisfies (28), and so prove the lemma.

5. In Lemma 3—20 we suppose that $\alpha, \beta, \bar{\beta}$ satisfy (16) and (19), and that (17) and (18) are satisfied for all integers $\xi \neq 0$ of $k(\theta)$, and we prove that $\alpha, \beta, \bar{\beta}$ are necessarily given by (20).

We summarize here certain identities satisfied by $\theta, \phi, \bar{\phi}$ and some numerical results.

We have

$$\theta^3 = \theta + 1,\ \theta^4 = \theta^2 + \theta,\ \theta^5 = \theta^2 + \theta + 1,$$

$$\theta^{-1} = \theta^2 - 1,\ \theta^{-2} = 1 + \theta - \theta^2,\ \theta^{-3} = \theta^2 - \theta,$$

$$\theta^{-4} = \theta - 1,\ \theta^{-5} = 2 - \theta^2,\ \theta^{-6} = 2\theta^2 - \theta - 2,\ \theta^{-7} = 1 + 2\theta - 2\theta^2,$$

and the same identities are satisfied by ϕ and $\bar{\phi}$. We have the symmetrical results

$$\theta\phi\bar{\phi} = 1,\ \phi\bar{\phi} + \bar{\phi}\theta + \theta\phi = -1,\ \theta + \phi + \bar{\phi} = 0.$$

The number ϕ is given by

$$\phi = -\tfrac{1}{2}\theta + i\sqrt{\tfrac{3}{4}\theta^2 - 1}.$$

Numerically,

$$\theta = 1 \cdot 3247\ 1795\ldots,\quad \phi = -0 \cdot 6623\ 5897\ldots + i\,0 \cdot 5622\ 7951\ldots;$$

also

$$\theta^2 = 1 \cdot 7548\ 7766\ldots,\quad \theta^{\frac{1}{2}} = 1 \cdot 1509\ 6392\ldots.$$

Using these results we obtain the following numerical values for the various powers of θ:

n	θ^n	n	θ^n
5	4.079595...	$-\frac{1}{8}$	0.868836...
4	3.079595...	-1	0.754877...
$\frac{7}{8}$	2.675666...	$-\frac{8}{8}$	0.655865...
3	2.324717...	-2	0.569840...
$\frac{5}{8}$	2.019800...	-3	0.430159...
2	1.754877...	-4	0.324717...
$\frac{3}{8}$	1.524702...	-5	0.245122...
1	1.324717...	-6	0.185037...
$\frac{1}{8}$	1.150963...	-7	0.139680...

Lemma 3.

$$\alpha \geqq \{\tfrac{8}{9}(1-\varepsilon)\}^{\frac{1}{4}}. \quad \ldots\ldots\ldots \quad (34)$$

Proof. By (17) we have

$$\left|\left(\xi-\frac{1}{\alpha}\right)\left(\eta-\frac{1}{\beta}\right)\left(\bar{\eta}-\frac{1}{\bar{\beta}}\right)\right| \geqq \frac{1-\varepsilon}{|\alpha\beta\bar{\beta}|},$$

for all integers ξ of $k(\theta)$. It follows from Lemma 2 that

$$\frac{1-\varepsilon}{|\alpha\beta\bar{\beta}|} \leqq \tfrac{9}{8}. \quad \ldots\ldots\ldots \quad (35)$$

So, using (19),

$$\alpha \geqq \{\alpha|\beta|^2\}^{\frac{1}{4}} \geqq \{\tfrac{8}{9}(1-\varepsilon)\}^{\frac{1}{4}},$$

which gives (34).

Lemma 4.

$$\alpha > 1+\theta^{-7}. \quad \ldots\ldots\ldots \quad (36)$$

Proof. Suppose, if possible, that $\alpha \leq 1+\theta^{-7}$. It is easy to verify that

$$1-\theta^{-7} = 2\theta^2-2\theta = 0\cdot 860\ldots < \{\tfrac{8}{9}(1-\varepsilon)\}^{\frac{1}{4}},$$

if ε is sufficiently small. So by (34) and our supposition, we have

$$1-\theta^{-7} < \alpha \leqq 1+\theta^{-7}.$$

Hence, using (17), with $\xi = 1$,

$$|(\beta-1)(\bar{\beta}-1)| \geqq \frac{1-\varepsilon}{|\alpha-1|} \geqq (1-\varepsilon)\,\theta^7,$$

so that

$$|\beta-1| \geqq \sqrt{(1-\varepsilon)\,\theta^7} > 2\cdot 6,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$1\cdot 6 < |\beta| \leqq \sqrt{\alpha} \leqq \sqrt{\{1+\theta^{-7}\}} < 1\cdot 1,$$

a contradiction. This proves the lemma.

Lemma 5.

$$\alpha > \theta + \theta^{-6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Proof. Suppose, if possible, that $\alpha \leq \theta + \theta^{-6}$. Now

$$1 + \theta^{-7} = 2 + 2\theta - 2\theta^2 = \theta - \theta^{-6}.$$

It follows by Lemma 4 that

$$|\alpha - \theta| \leqslant \theta^{-6}.$$

Hence, using (18), with $\xi = \theta^{-1}$,

$$|(\beta - \phi)(\bar{\beta} - \bar{\phi})| \geqslant (1-\varepsilon)\,\theta^6,$$

so that

$$|\beta - \phi| \geqslant \sqrt{(1-\varepsilon)}\,\theta^3 > 2 \cdot 3,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$\begin{aligned} 1 \cdot 4 &< 2 \cdot 3 - \theta^{-\frac{1}{2}} = 2 \cdot 3 - |\phi| \\ &\leqslant |\beta| \leqslant \sqrt{\alpha} \\ &\leqslant \sqrt{\{\theta + \theta^{-6}\}} = \sqrt{\{2\theta^2 - 2\}} < 1 \cdot 3, \end{aligned}$$

a contradiction. This proves the lemma.

Lemma 6.

$$\alpha > \theta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Proof. Suppose, if possible, that $\alpha \leq \theta^2$. By (18) with $\xi = \theta^{-1}$ and $\xi = \theta^{-2}$ we have

$$|(\alpha - \theta)(\beta - \phi)(\bar{\beta} - \bar{\phi})| \geqslant 1 - \varepsilon,$$

$$|(\alpha - \theta^2)(\beta - \phi^2)(\bar{\beta} - \bar{\phi}^2)| \geqslant 1 - \varepsilon.$$

Hence

$$|\beta - \phi|^2 + |\beta - \phi^2|^2 \geqslant \frac{1-\varepsilon}{|\alpha - \theta|} + \frac{1-\varepsilon}{|\alpha - \theta^2|} \quad . \quad . \quad . \quad . \quad (39)$$

But by Lemma 5 and our supposition

$$\theta < \alpha \leqslant \theta^2,$$

and so

$$\begin{aligned} &\frac{1}{|\alpha - \theta|} + \frac{1}{|\alpha - \theta^2|} \\ &\quad = \frac{\theta^2 - \theta}{(\alpha - \theta)(\theta^2 - \alpha)} \geqslant \frac{\theta^2 - \theta}{\{\frac{1}{2}(\alpha - \theta + \theta^2 - \alpha)\}^2} \\ &\quad = \frac{4}{\theta^2 - \theta} = 4\theta^3. \end{aligned}$$

Thus, if ε is sufficiently small, we have by (39)

$$|\beta - \phi|^2 + |\beta - \phi^2|^2 > 9 \cdot 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

We now use the identity

$$|2\beta - \eta_1 - \eta_2|^2 = 2|\beta - \eta_1|^2 + 2|\beta - \eta_2|^2 - |\eta_2 - \eta_1|^2. \quad . \quad . \quad (41)$$

with $\eta_1 = \phi$, $\eta_2 = \phi^2$. Using (41) and (40) we obtain

$$\left.\begin{aligned} |\beta| &\geqq \tfrac{1}{2}\{|2\beta-\phi-\phi^2|-|\phi+\phi^2|\} \\ &= \tfrac{1}{2}\{2|\beta-\phi|^2+2|\beta-\phi^2|^2-|\phi^2-\phi|^2\}^{\frac{1}{2}}-\tfrac{1}{2}|\phi+\phi^2| \\ &> \tfrac{1}{2}\{18\cdot 4-|\phi|^{-6}\}^{\frac{1}{2}}-\tfrac{1}{2}|\phi|^4 \\ &= \tfrac{1}{2}\{18\cdot 4-\theta^3\}^{\frac{1}{2}}-\tfrac{1}{2}\theta^{-2} \\ &> \tfrac{1}{2}\{18\cdot 4-2\cdot 4\}^{\frac{1}{2}}-0\cdot 3 = 1\cdot 7. \end{aligned}\right\} \quad (42)$$

But, by (19) and our supposition concerning α,

$$|\beta| \leqq |'\alpha \leqq \theta < 1\cdot 4,$$

contrary to (42). This contradiction proves the lemma.

Lemma 7.

$$\alpha > \theta^3 - \theta^{-6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Proof. Suppose, if possible, that $\alpha \leq \theta^3 - \theta^{-6}$. By (18) with $\xi = \theta^{-2}$ $\xi = \theta^{-3}$ we have

$$|(\alpha-\theta^2)(\beta-\phi^2)(\bar{\beta}-\bar{\phi}^2)| \geqq 1-\varepsilon,$$
$$|(\alpha-\theta^3)(\beta-\phi^3)(\bar{\beta}-\bar{\phi}^3)| \geqq 1-\varepsilon.$$

Hence

$$|\beta-\phi^2|^2+|\beta-\phi^3|^2 \geqq \frac{1-\varepsilon}{|\alpha-\theta^2|}+\frac{1-\varepsilon}{|\alpha-\theta^3|}. \quad . \quad . \quad . \quad (44)$$

But by Lemma 6 and our supposition

$$\theta^2 < \alpha < \theta^3,$$

and so

$$\begin{aligned} &\frac{1}{|\alpha-\theta^2|}+\frac{1}{|\alpha-\theta^3|} \\ &\quad = \frac{\theta^3-\theta^2}{(\alpha-\theta^2)(\theta^3-\alpha)} \\ &\quad \geqq \frac{\theta^3-\theta^2}{\{\frac{1}{2}(\alpha-\theta^2+\theta^3-\alpha)\}^2} = \frac{4}{\theta^3-\theta^2} = 4\theta^2. \end{aligned}$$

Thus, if ε is sufficiently small, we have, by (44),

$$|\beta-\phi^2|^2+|\beta-\phi^3|^2 > 7\cdot 01. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Using the identity (41) with $\eta_1 = \phi^2$, $\eta_2 = \phi^3$, and (45), we obtain

$$\left.\begin{aligned} |\beta| &\geqq \tfrac{1}{2}\{|2\beta-\phi^2-\phi^3|-|\phi^2+\phi^3|\} \\ &= \tfrac{1}{2}\{2|\beta-\phi^2|^2+2|\beta-\phi^3|^2-|\phi^3-\phi^2|^2\}^{\frac{1}{2}}-\tfrac{1}{2}|\phi^2+\phi^3| \\ &> \tfrac{1}{2}\{14\cdot 02-|\phi|^{-4}\}^{\frac{1}{2}}-\tfrac{1}{2}|\phi|^5 \\ &= \tfrac{1}{2}\{14\cdot 02-\theta^2\}^{\frac{1}{2}}-\tfrac{1}{2}\theta^{-\frac{1}{2}} \\ &> \tfrac{1}{2}\{14\cdot 02-1\cdot 76\}^{\frac{1}{2}}-0\cdot 25 \\ &> \tfrac{1}{2}\{12\cdot 25\}^{\frac{1}{2}}-0\cdot 25 = 1\cdot 5. \end{aligned}\right\} \quad (46)$$

But, by (19) and our supposition concerning α,

$$|\beta| \leqq \sqrt{\alpha} \leqq \sqrt{\{\theta^3 - \theta^{-6}\}} < \{2 \cdot 33 - 0 \cdot 18\}^{\frac{1}{2}} < 1 \cdot 5,$$

contrary to (46). This contradiction proves the lemma.

Lemma 8.

$$\alpha > \theta^3 + \theta^{-6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Proof. Suppose, if possible, that $\alpha \leq \theta^3 + \theta^{-6}$ Then, by Lemma 7,

$$|\alpha - \theta^3| \leqq \theta^{-6}.$$

Hence, using (18), with $\xi = \theta^{-3}$,

$$|(\beta - \phi^3)(\bar{\beta} - \bar{\phi}^3)| \geqq (1 - \varepsilon)\,\theta^6,$$

so that

$$|\beta - \phi^3| \geqq \theta^3 \sqrt{(1 - \varepsilon)} > 2 \cdot 32,$$

if ε is sufficiently small. Using this result and (19) we obtain

$$\begin{aligned} 1 \cdot 66 &< 2 \cdot 32 - \theta^{-\frac{3}{2}} = 2 \cdot 32 - |\phi^3| \\ &< |\beta - \phi^3| - |\phi^3| \leqq |\beta| \\ &\leqq \sqrt{\alpha} \leqq \sqrt{\{\theta^3 + \theta^{-6}\}} = \sqrt{\{2\,\theta^2 - 1\}} \\ &< \sqrt{\{2 \cdot 51\}} < 1 \cdot 6, \end{aligned}$$

a contradiction. This proves the lemma.

(*To be continued.*)

Mathematics. — *Absolutes in partially ordered groups.* By LADISLAS FUCHS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 29, 1949.)

In a lattice-ordered group G. BIRKHOFF [1]) has defined the *absolute* of an element a as the element $a \cup -a$, that is, the join of a and $-a$, and has shown that many of the properties of the common absolute value are valid for the new absolute. We now generalize further this notion to every partially ordered group. The basic idea is the observation that a lattice-ordered group plays the same role in the theory of partially ordered groups as a principal ideal ring among the rings. Therefore, if we consider sets instead of elements, viz., instead of the sole element $a \cup -a$, the collection of the elements greater than or equal to $a \cup -a$, then the concept of absolute (considered now as a set) may be extended to every partially ordered group. Indeed, it may be defined in terms of the upper bounds of certain elements depending on a.

Let us recapitulate the definition of an ordered group [2]). The group G, written additively, is called an *ordered group*, if a relation $\geqq$ is defined in G which satisfies the following postulates:

reflexive law: $a \geqq a$; . . . . . (1)

antisymmetric law: $a \geqq b$ *and* $b \geqq a$ *imply* $a = b$; . . . . (2)

transitive law: $a \geqq b$ *and* $b \geqq c$ *imply* $a \geqq c$; . . . . . (3)

homogeneity law: $a \geqq b$ *implies* $c + a + d \geqq c + b + d$ *for every* c, d *in* G: . . . . . (4)

the MOORE—SMITH *property*: *for every* a, b *there exists a third element* c *such that* $c \geqq a$ *and* $c \geqq b$. . . . . (5)

The MOORE—SMITH property is known to be equivalent to the assertion that [3])

each element of G *is the difference of two positive elements*. . . . . (6)

Let $U(x_1, \ldots, x_n) = U(x_i)$ and $L(x_1, \ldots, x_n) = L(x_i)$ denote the set of all upper bounds and all lower bounds, respectively, for the finite

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1) G. BIRKHOFF, Lattice-ordered groups, Ann. Math. **43**, 298—331 (1942). The notion of absolute is originally due to L. V. KANTOROVITSCH, Lineare halbgeordnete Räume, Mat. Sbornik, **44**, 121—168 (1937).

2) C. J. EVERETT and S. ULAM, On ordered groups, Trans. Am. Math. Soc. **57**, 208—216 (1945).

3) A. H. CLIFFORD, Partially ordered abelian groups, Ann. Math. **41**, 465—473 (1940). For the non-commutative case see BIRKHOFF, loc. cit.

number of elements $x_1, \ldots, x_n$ of G, that is, $u \in U(x_i)$ if and only if $u \geqq x_i$ for $i = 1, 2, \ldots, n$, and dually, $v \in L(x_i)$ if and only if $v \leqq x_i$ for $i = 1, 2, \ldots, n$. By (5) neither $U(x_i)$ nor $L(x_i)$ is empty.

There is an intrinsic connection between the sets $U(x_i)$ and $L(x_i)$, namely [4]),

$$U(x_i) = -L(-x_i). \quad \ldots \quad (7)$$

If $u \in U(x_i)$, that is, $u \geqq x_i$, then by making use of (4) one concludes $-u \leqq -x_i$, which shows that $-u \in L(-x_i)$, $u \in -L(-x_i)$; and vice-versa. (7) indicates that proving propositions for $L(x_i)$ is unnecessary if the corresponding one has been established for $U(x_i)$.

One can define addition between elements and sets U, L. By $a + U + b$ we shall understand the totality of the elements $a + u + b$ with $u \in U$. Similar definitions apply to $a + L + b$, $a - U + b$ and $a - L + b$. We now prove

$$a + U(x_i) + b = U(a + x_i + b). \quad \ldots \quad (8)$$

Assume $u \in U(x_i)$, i.e. $u \geqq x_i$ for every i. By (4) we have $a + u + b \geqq a + x_i + b$ whence $a + u + b \in U(a + x_i + b)$. Conversely, if $y \in U(a + x_i + b)$, $y \geqq a + x_i + b$, then $u = -a + y - b \geqq x_i$, so that $u \in U(x_i)$ and $y = a + u + b \in a + U(x_i) + b$, by definition. Dually we get

$$a + L(x_i) + b = L(a + x_i + b). \quad \ldots \quad (8a)$$

Another interesting and important law is

$$a - U(x_i) + b = L(a - x_i + b). \quad \ldots \quad (9)$$

Let $u \in U(x_i)$, $u \geqq x_i$, then by making use of (4), $a - u + b \leqq a - x_i + b$, that is, $a - u + b \in L(a - x_i + b)$. On the other hand, if $v \in L(a - x_i + b)$, $v \leqq a - x_i + b$, then $x_i \leqq b - v + a = u$, $u \in U(x_i)$ and hence $v = a - u + b \in a - U(x_i) + b$. The dual of (9) is

$$a - L(x_i) + b = U(a - x_i + b). \quad \ldots \quad (9a)$$

Putting $a = b = 0$ in (9) or in (9a) we reget (7). Further applying (9) to two elements and substituting a for x_1, b for x_2, one is led directly to the result:

$$a - U(a, b) + b = L(a, b) \quad \ldots \quad (10)$$

and dually,

$$a - L(a, b) + b = U(a, b). \quad \ldots \quad (10a)$$

By $U_1 + U_2$ is meant the totality of all elements of the form $u_1 + u_2$ where $u_1 \in U_1$, $u_2 \in U_2$. In similar ways are defined $L_1 + L_2$, $U - L$, $-L + U$, etc. It must be observed that an expression such as $U + L$ has no meaning, since the elements of the form $u + v$ ($u \in U$, $v \in L$) constitute the whole group. Indeed, if a is arbitrary in G by (6) we have $a = b - c$ with $b \geqq 0$, $c \geqq 0$. Choose an element d belonging to both U and $-L$; this is possible by (5). Then $b + d$ again belongs

[4]) $-L$ consists of the elements of the form $-v$ with $v \in L$.

to U and $-d\ -b$ again belongs to L, consequently, $U+L$ contains $(b+d)+(-d\ -c)=b-c=a$.

It is obvious that the addition of the U's is an associative operation $(U_1+U_2)+U_3=U_1+(U_2+U_3)$ and if G is commutative, it is commutative too.

Next we turn our attention to the proof of the relation:

$$U(x_i)+U(y_j)\subseteqq U(x_i+y_j). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

To prove (11), suppose $t\in U(x_i)$, $u\in U(y_j)$, that is, $t\geqq x_i$, $u\geqq y_j$, then $t+u\geqq x_i+u\geqq x_i+y_j$, therefore, $t+u$ belongs to $U(x_i+y_j)$, in fact. (8) states that in case there is only one x_i or only one y_j (11) is true with the sign of equality instead of $\subseteqq$. The same is the case even if the order is a lattice-order. However, in general, (11) is not true with $=$. For example, in the group G_2 of all two-dimensional vectors with integral components, $[m, n]$, let the order be defined by the axiom

$$[m, n]>[p, q] \text{ if and only if } m>p \text{ and } n>q.$$

Then $U_1=U([0,1],[1,0])$ and $U_2=U([1,2],[2,1])$ consist of vectors, both components of which are $\geqq 2$ and $\geqq 3$, respectively, so that U_1+U_2 contains the vectors with components $\geqq 5$. Nevertheless $U([1,3],[2,2],[2,2],[3,1])$ consists of vectors whose components are $\geqq 4$, i.e., it contains properly U_1+U_2. If in G_2 it were allowed the components to be arbitrary rational numbers, or, even real numbers, then we could at once convince ourselves that instead of (11) the sharper

$$U(x_i)+U(y_j)=U(x_i+y_j) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

holds. Orders satisfying (12) are called *distributive*. The last example shows that lattice-order is not necessary for distributivity; but it suffices.

By the positive part of a is meant a set of elements of G such that $a^+=U(a, 0)$. $a^-=L(a, 0)$ is called the negative part of a. From (7) we get

$$a^-=L(a,0)=-U(-a,0)=-(-a)^+. \quad . \quad . \quad . \quad . \quad (13)$$

When we apply (10) and (10a) to $a=0$, then to $b=0$, we are led to the result:

$$a-a^+=-a^++a=a^- \text{ and } a-a^-=-a^-+a=a^+. \quad . \quad (14)$$

Hence it follows, immediately, that a commutes with the sets $-a^+$ and $-a^-$. It is also of importance that

$$a^+ \text{ and } -a^-=(-a)^+ \text{ are permutable: } a^++(-a)^+=(-a)^++a^+. \quad (15)$$

In order to verify this assertion, use (10a) repeatedly, then we get

$$\begin{aligned} a^++(-a)^+&=U(a,0)+U(-a,0)=(-L(0,a)+a)+(-a-L(-a,0))\\ &=-L(0,a)-L(-a,0)=U(-a,0)+U(a,0)=(-a)^++a^+, \end{aligned}$$

as we wished to prove.

Let us now turn to the definition of the "absolute". The definition may be given in two different manners.

I. *Absolute of the first type* (1-*absolute*) *is defined by*

$$|a| = a^+ - a^- = a^+ + (-a)^+. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(15) shows that an equivalent definition of $|a|$ would be $|a| = (-a)^+ + a^+$.

II. *By absolute of the second type* (2-*absolute*) *we mean*

$$\|a\| = U(a, -a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The connection between the two types of absolute is given by

$$|a| \subseteq \|a\| \quad \textit{for every } a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Indeed, using (11) we obtain

$$|a| = U(a,0) + U(-a,0) \subseteq U(0, a, -a, 0) \subseteq U(a, -a) = \|a\|. \quad (18')$$

In general, the sign $\subseteq$ can not be sharpened to the sign of equality; e.g. in the group G_2 considered above

$$|[2, -1]| \subset \|[2, -1]\|$$

for the first consists of vectors $[m, n]$ with $m \geqq 4$, $n \geqq 3$, while the second of those with $m \geqq 3$, $n \geqq 2$. But, whenever the order defined in G is distributive as well as *normal* in the sense that

$$na = a + a + \ldots + a \geqq 0 \quad \textit{for some positive integer } n \textit{ implies } a \geqq 0, \quad (19)$$

then the 1-absolute and the 2-absolute are equal. In proving the equality, first we show that normality implies $U(a, -a, 0) = U(a, -a)$. It is clearly enough to prove that $U(a, -a) \subseteq U(a, -a, 0)$, or, in other words, $b \geqq a$, $b \geqq -a$ imply $b \geqq 0$. From $b \geqq a$, $b \geqq -a$ it follows by (4) and (3) that $2b \geqq b + a \geqq -a + a = 0$, and hence by normality we conclude that $b \geqq 0$, as stated. Now we have, by distributivity, the sign of equality instead of $\subseteq$ in (18').

Since a lattice-ordered group is distributive and normal, therefore in every lattice-ordered group both absolutes coincide.

Next we exhibit the properties of the absolute.

$$|0| = G^+, \quad |a| \subset G^+ \quad \textit{if } a \neq 0. \ [5] \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$|0| = G^+$ and $|a| \subseteq G^+$ are trivial consequences of the definition (16). Further it is evident that $|a| = U(-a, 0) + U(a, 0) = G^+$ if and only if $U(-a, 0) = G^+$ and $U(a, 0) = G^+$. These imply $-a \leqq 0$ and $a \leqq 0$, respectively, so that $a = 0$.

$$\|0\| = G^+ \textit{ and if the order is normal, } \|a\| \subset G^+ \textit{ when } a \neq 0. \quad (21)$$

As we have seen in the proof above, normality implies $U(a, -a, 0) = U(a, -a)$, so that $\|a\|$ contains only positive elements, $\|a\| \subseteq G^+$. Further $\|a\| = U(a, -a, 0) = G^+$ implies $a \leqq 0$, $-a \leqq 0$; therefore $a = 0$.

$$|-a| = |a| \textit{ is a consequence of } (15), (16). \quad . \quad . \quad . \quad (22)$$

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[5] $G+$ denotes the set of all positive elements (together with 0), that is, $G+ = U(0)$.

$\|-a\|=\|a\|$ *is a consequence of the symmetrical definition of* $\|a\|$. (23)

$$|a-b|=U(a,b)-L(a,b) \text{ for every pair } a,b \text{ in } G. \quad . \quad . \quad (24)$$

This gives for $b=0$ the definition (16). On using (8) we obtain

$$a-b|=U(a-b,0)-L(a-b,0)=\{U(a,b)-b\}-\{L(a,b)-b\}= U(a,b)-L(a,b).$$

The 2-absolute fails to have the corresponding property.

$$n\,|a| \subseteq |na| \text{ for every positive integer } n. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Obviously,

$$|n|a|=n\{U(a,0)+U(-a,0)\}=U(a,0)+U(-a,0)+U(a,0)+U(-a,0)+\ldots.$$

Since the sets $U(a,0)$ and $U(-a,0)$ are by (15) permutable, we get $n\,|a|=nU(a,0)+nU(-a,0)$. Hence, in view of (11) we infer that $n\,|a| \subseteq U(na,\ldots,a,0)+U(-na,\ldots,-a,0) \subseteq U(na,0)+U(-na,0)=|na|$, q.e.d. From the proof it is clear that distributivity does not suffice for proving the equality in (25).

$$\text{For every positive integer } n \text{ we have } n\,\|a\| \subseteq \|na\|, \quad . \quad . \quad . \quad (26)$$

since $n\,\|a\|=nU(a,-a) \subseteq U(na,-na)=\|na\|$.

In case G is a *commutative* group, we can in addition prove the analogue of the triangular inequality; the sign of inequality must hold, of course, in the opposite sense:

$$|a+b| \supseteq |a|+|b|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

For, $|a|+|b|=U(a,0)-L(a,0)+U(b,0)-L(b,0)=U(a,0)+U(b,0)-L(a,0)-L(b,0) \subseteq U(a+b,0)-L(a+b,0)=|a+b|$.

$$\|a\|+\|b\|=U(a,-a)+U(b,-b) \subseteq U(a+b,-a-b)= U(a+b,-b-a)=\|a+b\| \quad (28)$$

provided that G is commutative.

Mathematics. — *The product of the minima and the determinant of a set.* By C. A. ROGERS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 26, 1949.)

1. We use X to denote the general point with coordinates $(x_1, \ldots, x_n)$ in n-dimensional space and we use a vector notation. If

$$A_1 = (a_{11}, \ldots, a_{n1}), \ldots, A_n = (a_{1n}, \ldots, a_{nn})$$

are any n points, such that the determinant $\triangle$ of the matrix (a_{rs}) is not zero, the set Λ of all points X of the form

$$X = u_1 A_1 + \ldots + u_n A_n,$$

where $u_1, \ldots, u_n$ take all integral values, is said to be the lattice with determinant $d(\Lambda) = |\triangle|$ generated by $A_1, \ldots, A_n$.

A well known theorem of MINKOWSKI asserts that, if K is any n-dimensional convex body with the origin $O = (0, \ldots, 0)$ as centre, and if Λ is any lattice with no point other than O in K, then the volume $V(K)$ of K satisfies the inequality

$$V(K) \leqq 2^n d(\Lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The first minimum $\mu_1 = \mu_1(S, \Lambda)$ of any set S for a lattice Λ is defined to be the lower bound of the positive numbers μ such that the set μS of all points μX with X in S contains a point different from O of Λ. It is clear that, with this definition, MINKOWSKI's theorem can be restated in the following form. If K is any convex body with the origin O as centre and with volume $V(K)$, then for every lattice Λ

$$\mu_1^n V(K) \leqq 2^n d(\Lambda), \qquad (\mu_1 = \mu_1(K, \Lambda)). \quad . \quad . \quad . \quad . \quad (2)$$

MINKOWSKI [1]) refined this inequality, by associating with the convex body K and the lattice Λ a succession of minima $\mu_1 = \mu_1(K, \Lambda), \ldots, \mu_n = \mu_n(K, \Lambda)$ satisfying $0 < \mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_n$, and by proving that

$$\mu_1 \ldots \mu_n V(K) \leqq 2^n d(\Lambda), \qquad (\mu_i = \mu_i(K, \Lambda)). \quad . \quad . \quad . \quad (3)$$

In this paper we obtain an analogous inequality for the product of the successive minima and the critical determinant of an arbitrary set.

There are various ways of defining the successive minima of an arbitrary set. The following is a natural generalization of MINKOWSKI's definition.

Definition [2]) **(1).** *The k-th minimum $\mu_k = \mu_k(S, \Lambda)$ of a set S for a*

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[1]) H. MINKOWSKI, *Geometrie der Zahlen*, (Berlin, 1910), Kapitel 5. For a simple proof see H. DAVENPORT, *Quart. J. of Math.*, **10**, 119—121 (1939).

[2]) Our notation is consistent with that of V. JARNIK, *Časopis pro pěstování matematiky a fysiky*, **73**, 9—15 (1948) (in English) and consequently differs from that of other papers on this subject.

lattice Λ is defined to be the lower bound of the positive numbers μ such that the set μS contains at least k linearly independent points of Λ. By convention $\mu_k = +\infty$ if there are no such numbers μ.

It is clear from this definition that

$$0 \leqq \mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_n. \quad \ldots\ldots\ldots \quad (4)$$

The *critical determinant* $\Delta(S)$ of a set S is defined to be *the lower bound of the determinants of the lattices Λ with no point different from O in S.* [Note that this definition of the critical determinant does not coincide with that given by MAHLER in *Revista, Tucuman*, serie A, 5 (1946), 113—124, and in the *Canadian Journal of Maths.*, 1 (1949). The definitions coincide when S is open, and they are equivalent to that of MAHLER, *Proc. Royal Soc.* (A), 187 (1946), 151—187, when S is a star body [3].] With this definition it is easy to see that, for any [4] set S and any lattice Λ

$$\mu_1^n \Delta(S) \leqq d(\Lambda), \qquad (\mu_1 = \mu_1(S, \Lambda)). \quad \ldots\ldots \quad (5)$$

Further MINKOWSKI's result (1) can be restated in the form that

$$V(K) \leqq 2^n \Delta(K), \quad \ldots\ldots\ldots \quad (6)$$

for any convex body K with O as centre.

When (6) is satisfied with equality, for example when K is an n-dimensional cube with O as centre, it follows by MINKOWSKI's result (3) that

$$\mu_1 \ldots \mu_n \Delta(K) \leqq d(\Lambda), \qquad (\mu_l = \mu_l(K, \Lambda)) \quad \ldots\ldots \quad (7)$$

This inequality (7) has also been proved by MINKOWSKI [5] when K is an n-dimensional ellipsoid with O as centre. Again (7) is true if K is any 2-dimensional convex domain symmetrical in O. It has been conjectured that (7) is valid for any convex body K symmetrical in O; but the best general result [6] known is that, for any convex body K symmetrical in O,

$$\mu_1^{n-1} \mu_n \Delta(K) \leqq d(\Lambda), \qquad (\mu_l = \mu_l(K, \Lambda)) \quad \ldots\ldots \quad (8)$$

The first general inequality for the successive minima of non-convex sets was obtained by JARNIK [7]. Using a different definition he established an inequality for the product of the successive minima of the difference

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[3]) For the definition of a star body see MAHLER, *loc. cit.* It is not very difficult to prove that the following geometrical definition is equivalent to MAHLER's definition. The set S is a star body, if it is closed and if, for every λ with $-1 < \lambda < 1$ and for every point X of S, the point λX is an inner point of S.

[4]) If $\Delta(S) = +\infty$, we interpret (5) as implying that $\mu_1 = 0$; and, if $\mu_1 = +\infty$, we interpret (5) as implying that $\Delta(S) = 0$.

[5]) H. MINKOWSKI, *loc. cit.* § 51.

[6]) This can be proved by a simple modification of DAVENPORT's proof of MINKOWSKI's result (3), see DAVENPORT, *loc. cit.*

[7]) V. JARNIK, *Věstnik Královské České Společnosti Nauk*, (Praha, 1941), in Czech with a German summary.

set of a set S and the inner JORDAN content of S. This result of JARNIK was improved first by JARNIK and KNICHAL [8]) and then by myself [9]). The first direct inequality for the product of the successive minima of a set and its critical determinant was obtained by CHABAUTY [10]) who used JARNIK's method to prove that

$$\mu_1 \ldots \mu_n \triangle(S) \leqq 2^{n-1-(1/2)-\ldots-(1/n)}\, d(\Lambda), \qquad (\mu_i = \mu_i(S, \Lambda)) \quad . \quad (9)$$

for all [11]) star sets [12]) S. Before the publication of CHABAUTY's result MAHLER obtained an inequality of this form but with the larger constant $n!$ on the right hand side [13]). Dr. MAHLER was kind enough to let me see his manuscript. It then became plain to me that the inequality

$$\mu_1 \ldots \mu_n \triangle(S) \leqq 2^{n-1}\, d(\Lambda), \qquad (\mu_i = \mu_i(S, \Lambda)), \quad . \quad . \quad . \quad (10)$$

for all star sets S was implicit in JARNIK's paper (*loc. cit.*, 1941); and that the inequality

$$\mu_1 \ldots \mu_n \triangle(S) \leqq 2^{(n-1)/2}\, d(\Lambda), \qquad (\mu_i = \mu_i(S, \Lambda)), \quad . \quad . \quad . \quad (11)$$

for all sets S, was implicit in my paper (*loc. cit.*). The object of the present paper is to give a simple proof of this inequality (11). Dr. MAHLER, in a paper following this, shows by an ingenious n-dimensional example that the constant $2^{(n-1)/2}$ in (11) is the best possible, even for bounded star bodies.

In the next sections we give a proof of the inequality (11) in a slightly refined form. In the last section we state some results involving a second type of successive minima.

This paper owes its existence and its simplicity to many suggestions I have received at various times from Professors DAVENPORT and JARNIK and Drs. ESTERMANN and MAHLER. I am most grateful to these mathematicians.

2. **Theorem 1.** *Let S be any set and let Λ be any lattice. Suppose that, for some positive number μ and some positive integers $m_1, \ldots, m_n$,*

(a) m_{k+1} is an integral multiple of m_k, if $1 \leq k < n$ and

(b) $\mu m_k \leq \mu_k = \mu_k(S, \Lambda)$, if $1 \leq k \leq n$.

Then $\triangle(S) < +\infty$ and

$$\mu^n m_1 \ldots m_n \triangle(S) \leqq d(\Lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

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[8]) V. JARNIK and V. KNICHAL, *Rozpravy II třídy České Akademie*, 53 (1943), Cislo 43, in Czech. For a translation into French see *Bulletin international de l'Académie tchèque des Sciences*, 47 (1946), N°. 18.

[9]) C. A. ROGERS, *Proc. London Math. Soc. (2)*, in the press.

[10]) C. CHABAUTY, *Comptes rendus*, 227, 747—749 (1948).

[11]) If $\triangle(S) = +\infty$, we interpret (9) as implying that $\mu_1 = 0$; and, if $\mu_n = +\infty$, we interpret (9) as implying that $\triangle(S) = 0$.

[12]) The set S is a star set if, for every point X of S and for all λ with $0 \leq \lambda \leq 1$, the point λX is in S.

[13]) Both CHABAUTY and MAHLER obtained their results without knowledge of my *Proc. London Math. Soc.* paper; had either of them been able to study my paper they could not but have obtained the inequality (11).

Further, if S is a bounded star body and $\mu m_n < \mu_n$, then (12) is satisfied with strict inequality.

Proof [14]. As $0 < \mu m_1 \leq \mu_1$ it is clear that $\Delta(S) < +\infty$. Let ν be any number satisfying $0 < \nu < \mu$. Let a_k be the smallest linear manifold (or the minimal subspace) containing O and the points of Λ in the set $\nu m_k S$; so that a_k is the set of all points X of the form

$$X = \xi_1 X_1 + \ldots + \xi_r X_r$$

where $\xi_1, \ldots, \xi_r$ are real and $X_1, \ldots, X_r$ are points of Λ in $\nu m_k S$. Then, by the condition (b) and the definition of μ_k, the set a_k is of dimension $d(k)$, say, less than k. Now by the condition (a) it is clear that, if P is a point of Λ in $\nu m_k S$, then $(m_{k+1}/m_k)P$ is a point of Λ in $\nu m_{k+1} S$. Hence a_{k+1} contains a_k if $1 \leq k < n$. Consequently we can choose a set of lattice points $P_1, \ldots, P_n$ generating Λ and such that a_k is the subspace generated by $O, P_1, \ldots, P_{d(k)}$ for $k = 1, \ldots, n$.

Consider the lattice Λ' generated by the points

$$\frac{1}{\nu m_1} P_1, \ldots, \frac{1}{\nu m_n} P_n.$$

Suppose, if possible, that X is a point other than O of Λ' in S. Then

$$X = \frac{u_1}{\nu m_1} P_1 + \ldots + \frac{u_n}{\nu m_n} P_n,$$

for some integers $u_1, \ldots, u_n$, not all zero. Let k be the integer with $1 \leq k \leq n$ such that

$$u_k \neq 0, \; u_{k+1} = 0, \ldots, u_n = 0.$$

By the condition (a) it is clear that the point

$$\nu m_k X = \frac{u_1 m_k}{m_1} P_1 + \ldots + \frac{u_k m_k}{m_k} P_k$$

is a point common to $\nu m_k S$ and Λ. But, since $u_k \neq 0$, it is clear that the point $\nu m_k X$ does not lie in the subspace generated by $O, P_1, \ldots, P_{k-1}$ and so does not lie in a_k. This is contrary to the definition of a_k. Consequently there is no point X different from O of Λ' in S. Hence

$$\Delta(S) \leqslant d(\Lambda') = \frac{1}{\nu^n m_1 \ldots m_n} d(\Lambda).$$

As ν may be arbitrarily close to μ this proves (12).

Now suppose that S is a bounded star body and that $\mu m_n < \mu_n$. In this case we take $\nu = \mu$ and take a_k to be the linear subspace generated by O and the points of Λ which are *inner* points of $\mu m_k S$. The lattice Λ' is

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[14]) This proof is based on V. JARNIK and V. KNICHAL, *loc. cit.*, 1943, Důkaz věty 1, or see V. JARNIK and V. KNICHAL, *loc. cit.*, 1946, Démonstration du théorème 1.

constructed as before, and the same argument now leads to the conclusion that there is no point X other than O of Λ' in the *interior* of S.

Let β_n be the linear subspace generated by O and the points of Λ which are points of $\mu m_n S$. Then β_n contains α_n and since $\mu m_n < \mu_n$ the dimension of β_n is less than n. We may thus suppose that β_n is contained in the $(n-1)$-dimensional space γ_n generated by $O, P_1, \ldots, P_{n-1}$. It follows by the argument used above that, if X is any point of Λ' in S, then X is in γ_n.

We have now proved that there is no point other than O of Λ' in the interior of S and that there are at most $n-1$ linearly independent points of Λ' on the boundary of S. As S is a bounded star body, it follows by a result of MAHLER [15]) that Λ' is not a critical lattice of S. Consequently by MAHLER's definition of a critical lattice

$$\triangle(S) < d(\Lambda') = \frac{1}{\mu^n m_1 \ldots m_n} d(\Lambda).$$

This completes the proof of the theorem.

3. **Theorem 2.** *Suppose that* $\mu_1, \ldots, \mu_n$ *are any numbers satisfying*

$$0 < \mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then there exist a positive number μ *and positive integers* $m_1, \ldots, m_n$, *such that*

(a) m_{k+1} *is an integral multiple of* m_k, *if* $1 \leq k < n$,

(b) $\mu m_k \leq \mu_k$, *if* $1 \leq k \leq n$, *and* (c)

$$\mu_1 \ldots \mu_n \leqq 2^{(n-1)/2} \mu^n m_1 \ldots m_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Further $\mu, m_1, \ldots, m_n$ *may be chosen to ensure that either* (14) *is satisfied with strict inequality or* $\mu m_n < \mu_n$.

Proof [16]). Write

$$\delta_k = \delta(k) = \log_2 \mu_k, \qquad k = 1, \ldots, n \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Let $r(x)$ be defined by the equation $r(x) = x - [x]$, where $[x]$ denotes as usual the integral part of x. Consider the sums

$$\sum_{k=1}^{n} r(\delta_k - \delta_h), \qquad h = 1, \ldots, n.$$

Since

$$r(\delta_k - \delta_h) + r(\delta_h - \delta_k) = 0$$

if $\delta_k - \delta_h$ is an integer, and

$$r(\delta_k - \delta_h) + r(\delta_h - \delta_k) = 1$$

otherwise, it is clear that

$$\sum_{h=1}^{n} \sum_{k=1}^{n} r(\delta_k - \delta_h) \leqq \tfrac{1}{2} n(n-1).$$

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[15]) K. MAHLER, Proc. Royal Soc. A, **187** (1946), 151—187 (162), Th. 11.

[16]) This proof is based on the proof of Theorem 2 of C. A. ROGERS, *loc. cit.*

Hence we can choose a number δ such that

$$\sum_{k=1}^{n} r(\delta_k - \delta) \leqq \tfrac{1}{2}(n-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We can also, by subtracting a sufficiently large integer from δ, ensure that $\delta \leq \delta_1$. Then by (13) and (15)

$$\delta \leqq \delta_1 \leqq \delta_2 \leqq \ldots \leqq \delta_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Write

$$\left.\begin{aligned} q_k = q(k) = [\delta_k - \delta], \quad & \text{for } k = 1, \ldots, n, \\ m_k = 2^{q(k)}, \quad & \text{for } k = 1, \ldots, n, \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

and $\mu = 2^{\delta}$. Then, by (17) and (18),

$$0 \leqq q_1 \leqq q_2 \leqq \ldots \leqq q_n,$$

so that m_{k+1} is an integral multiple of the integer m_k, for $k = 1, \ldots, n-1$. Also using (18), as

$$q_k = q(k) \leqq \delta_k - \delta, \text{ for } k = 1, \ldots, n,$$

we have

$$\mu m_k = 2^{q(k)+\delta} \leqq 2^{\delta(k)} = \mu_k, \text{ for } k = 1, \ldots, n,$$

by (15). Further, by (18), (15), the definition of $r(x)$, and (16), we have

$$\left.\begin{aligned} &\log_2(\mu_1 \ldots \mu_n) - \log_2(\mu^n m_1 \ldots m_n) \\ &\quad = \sum_{k=1}^{n} \delta_k - n\delta - \sum_{k=1}^{n} [\delta_k - \delta] = \sum_{k=1}^{n} r(\delta_k - \delta) \leqq \tfrac{1}{2}(n-1). \end{aligned}\right\} \quad (19)$$

Thus (14) is satisfied and we have found a positive number μ and positive integers $m_1, \ldots, m_n$ satisfying the conditions (a), (b) and (c).

We have still to prove the last clause of the theorem. It is clear that, unless

$$\sum_{k=1}^{n} r(\delta_k - \delta_h) = \tfrac{1}{2}(n-1), \text{ for } h = 1, \ldots, n, \quad . \quad . \quad . \quad (20)$$

we can choose δ to satisfy (16) with strict inequality and can deduce that (19) and (14) are satisfied with strict inequality. When (20) is satisfied the number δ satisfying (16) and (17) may be taken to be δ_1. Then, if μm_n were equal to μ_n, the number $\delta_n - \delta_1 = \delta_n - \delta$ would be equal to the integer q_n. This would imply that

$$\sum_{h=1}^{n} \sum_{k=1}^{n} r(\delta_k - \delta_h) \leqq \tfrac{1}{2} n(n-1) - 1,$$

contrary to (20). Hence when (20) is satisfied we can ensure that $\mu m_n < \mu_n$ by taking $\delta = \delta_1$. This completes the proof of the theorem.

4. **Theorem 3.** *Let S be any set and let Λ be any lattice. Suppose that $\mu_1(S,\Lambda) > 0$ and that $\Delta(S) > 0$. Then $\mu_n(S,\Lambda) < +\infty$, $\Delta(S) < +\infty$ and* [17])

$$\mu_1 \ldots \mu_n \Delta(S) \leqq 2^{(n-1)/2} d(\Lambda), \qquad (\mu_i = \mu_i(S,\Lambda)). \quad . \quad . \quad (21)$$

Further, if S is a bounded star body, (21) is satisfied with strict inequality.

Proof. Since $\mu_1 > 0$ it is clear that $\Delta(S) < +\infty$. Suppose, if possible, that $\mu_n = +\infty$. Write $\mu = \mu_1$ and

$$m_1 = m_2 = \ldots = m_{n-1} = 1, \qquad m_n = N,$$

where N is a large positive integer. Then by Theorem 1,

$$\mu_1^n N \Delta(S) \leqq d(\Lambda).$$

As $\mu_1 > 0$ and as this is true for arbitrarily large values of N, it follows that $\Delta(S) = 0$. This is contrary to our hypothesis and consequently $\mu_n < +\infty$.

Now $\mu_1, \ldots, \mu_n$ are finite numbers satisfying the inequalities (13). Let μ, $m_1, \ldots, m_n$ be the corresponding numbers whose existence is asserted in Theorem 2. Then

$$\mu_1 \ldots \mu_n \leqq 2^{(n-1)/2} \mu^n m_1 \ldots m_n, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and, since the conditions of Theorem 1 are satisfied,

$$\mu^n m_1 \ldots m_n \Delta(S) \leqq d(\Lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Combining (22) and (23) we obtain (21).

It is clear from the last clause of Theorem 2 that we have strict inequality in (22) and (21) unless $\mu m_n < \mu_n$. But, when S is a bounded star body and $\mu m_n < \mu_n$, it follows from the last clause of Theorem 1 that we have strict inequality in (23) and (21). Hence (21) is satisfied with strict inequality when S is a bounded star body. This completes the proof of the theorem.

We now show how the main result of my previous paper can be deduced from Theorem 3. We use $\mathcal{D}S$ to denote the difference set of S, i.e. the set of all points of the form $X - Y$ where X and Y are in S. It has been essentially shown by BLICHFELDT [18]) that, for any measurable set S with a finite LEBESGUE measure $V(S)$,

$$V(S) \leqq \Delta(\mathcal{D}S); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and that, if S is measurable and has an infinite LEBESGUE measure, then $\Delta(\mathcal{D}S)$ is infinite. It now follows by Theorem 3 that, for any measurable set S with LEBESGUE measure $V(S)$, and for any lattice Λ,

$$\mu_1 \ldots \mu_n V(S) \leqq 2^{(n-1)/2} d(\Lambda), \qquad (\mu_i = \mu_i(\mathcal{D}S, \Lambda)); \quad . \quad . \quad (25)$$

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[17]) Note that the theorem implies that $\mu_1 = 0$ if $\Delta(S) = +\infty$, and that $\Delta(S) = 0$ if $\mu_n = +\infty$.

[18]) H. F. BLICHFELDT, *Trans. Amer. Math. Soc.*, **15** (1914), 227—235, does not work with the LEBESGUE measure. For a general proof see FENCHEL, *Acta Arithmetica*, **2** 230—241 (240—241) (1937).

with the interpretations that $\mu_1 = 0$, if $V(S) = +\infty$, and that $V(S) = 0$, if $\mu_n = +\infty$. This is the main result of my previous paper, in a different notation.

4. In this section we introduce a second set of successive minima.

Definition (2). *The k-th minimum $\nu_k = \nu_k(S, \Lambda)$ of a set S for a lattice Λ is defined to be the upper bound of the values of ν such that the set νS contains less than k linearly independent points of Λ.*

Clearly

$$0 \leqq \nu_1 \leqq \nu_2 \leqq \ldots \leqq \nu_n \leqq +\infty,$$

and comparing the Definitions (1) and (2)

$$\mu_1 \leqq \nu_1, \ldots, \mu_n \leqq \nu_n.$$

Using this definition (but with another notation) I have proved (*loc cit.* Th. 3) that

$$\mu_1 \ldots \mu_n \frac{\nu_k}{\mu_k} V(S) \leqq 2^{n-1} d(\Lambda), \text{ for } k = 1, \ldots, n, \quad . \quad . \quad (26)$$

and that

$$(\mu_1 \ldots \mu_n)^{1-(1/n)} (\nu_1 \ldots \nu_n)^{1/n} V(S) \leqq 2^{n-1} d(\Lambda), \quad . \quad . \quad . \quad (27)$$

where

$$\mu_l = \mu_l(\mathfrak{D} S, \Lambda), \qquad \nu_l = \nu_l(\mathfrak{D} S, \Lambda),$$

for every measurable set S with a finite positive LEBESGUE measure. JARNIK [19]) has shown that, in a certain sense, these results are of the best possible nature.

Just as these results can be obtained by a simple modification of the method of proving (25), the following results can be obtained by a simple modification of the method used above for proving (21).

Theorem 4. *Let S be any set and let Λ be any lattice. Suppose that $\mu_1(S, \Lambda) > 0$ and that $\Delta(S) > 0$. Then $\nu_n(S, \Lambda) < +\infty$, $\Delta(S) < +\infty$, and*

$$\mu_1 \ldots \mu_n \frac{\nu_k}{\mu_k} \Delta(S) \leqq 2^{n-1} d(\Lambda), \text{ for } k = 1, \ldots, n, \quad . \quad . \quad (28)$$

and

$$(\mu_1 \ldots \mu_n)^{1-(1/n)} (\nu_1 \ldots \nu_n)^{1/n} \Delta(S) \leqq 2^{n-1} d(\Lambda), \quad . \quad . \quad . \quad (29)$$

where

$$\mu_l = \mu_l(S, \Lambda), \qquad \nu_l = \nu_l(S, \Lambda).$$

Note that the inequalities (26) and (27) are consequences of the inequality (24) and the inequalities (28) and (29) with S replaced by $\mathfrak{D} S$. Further it follows from (24) and JARNIK's work that, in a certain sense, (28) and (29) are of the best possible nature.

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[19]) V. JARNIK, *loc. cit.*, (1948).

Mathematics. — *On the uniform distribution modulo 1 of lacunary sequences.* By P. ERDÖS and J. F. KOKSMA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 26, 1949.)

§ 1. As is well known one calls the sequence of real numbers $u_1, u_2, \ldots$ uniformly distributed modulo 1, if the number N' of those among the numbers

$$u_1 - [u_1],\ u_2 - [u_2], \ldots, u_N - [u_N],$$

which fall into an arbitrarily given part $\alpha \leqq u < \beta$ of the unit interval $0 \leqq u < 1$ satisfies the condition

$$\frac{N'}{N} \to \beta - \alpha, \text{ if } N \to \infty.$$

The difference $\left|\frac{N'}{N} - (\beta - \alpha)\right|$ is always $\leqq 1$ and for fixed $N \geqq 1$ one calls its upper bound, (if (α, β) is supposed to run through all couples with $0 \leqq \alpha < \beta \leqq 1$), the discrepancy $D(N)$ of the sequence. If

$$ND(N) = o(N), \quad \ldots\ldots\ldots \quad (1)$$

it is trivial that the sequence is uniformly distributed modulo 1 and as was proved by WEYL [1]), inversively (1) is a consequence of the distribution modulo 1, defined above.

One gets an interesting special case when putting

$$u_n = \theta \lambda_n \qquad (n = 1, 2, \ldots) \ldots\ldots\ldots \quad (2)$$

where

$$\lambda_1 < \lambda_2 < \ldots \ldots\ldots\ldots\ldots \quad (3)$$

denotes an increasing sequence of integers. FATOU [1]) already proved that such a sequence is everywhere dense modulo 1 in the unit interval for almost all values of θ, provided that the sequence (3) is lacunary, i.e. that for some positive constant δ

$$\lambda_{n+1} \geqq (1 + \delta)\lambda_n \qquad (n = 1, 2, \ldots). \ldots\ldots \quad (4)$$

HARDY–LITTLEWOOD [1]) and WEYL [1]) proved that for each sequence of integers (3) the sequence of numbers (2) is uniformly distributed modulo 1 for almost all θ. Hence for such sequences (1) holds. FOWLER [1]), KOKSMA [1]) and DREWES [2]) deduced improvements of (1). In the special case

$$\lambda_n = 2^n,$$

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[1]) References in "Diophantische Approximationen", Erg. d. Math. IV, 4 (1936) by J. F. KOKSMA (Kap. VIII and IX).

[2]) A. DREWES, Diophantische Benaderingsproblemen, Thesis Free University, Amsterdam (1945).

the problem is equivalent to the question how the digits 0, 1 are distributed in the dyadic expansion of θ. Here KHINTCHINE [1]) proved very sharp results.

Generally speaking, the problem is somewhat easier to handle for lacunary sequences (4) than in the general case (3). In this paper we consider the case of *lacunary* sequences of numbers

$$u_n = f(n, \theta),$$

which form a generalisation of the sequences defined by (2). The method used in this paper leads to great difficulties, if one tries to apply it in the general case. In a following paper we treat the *general* case with an other method, which in the specialised cases which are considered in the present paper would give a slightly less sharp result than we deduce here.

§ 2. In this paper we prove a general theorem in which as a special case is contained the following

Theorem 1. *Let δ denote an arbitrary positive constant and $\omega(n)$ a positive increasing function of $n = 1, 2, \ldots$ with $\omega(n) \to \infty$, if $n \to \infty$. Then for any sequence of positive numbers $\lambda_1, \lambda_2, \ldots$, which satisfy (4), the discrepancy $D(N)$ of the sequence (2) satisfies the inequality*

$$ND(N) = o\left(N^{\frac{1}{2}} \log^{\frac{3}{2}} N (\log\log N)^{\frac{1}{2}} \omega(N)\right) \quad . \quad . \quad . \quad . \quad (5)$$

for almost all θ.

This estimate is sharper than all known results. The exponent $\frac{1}{2}$ in the factor $N^{\frac{1}{2}}$ cannot be improved, as KHINTCHINE proved that in the special case $\lambda_n = 2^n$, we have

$$ND(N) = \Omega\left(N^{\frac{1}{2}} \sqrt{\log\log N}\right).$$

Another application of our theorem is the following

Theorem 2. *For almost all values of $\theta \geqq 1$ the discrepancy of the sequence*

$$\theta, \theta^2, \theta^3, \ldots$$

satisfies the inequality (5), if $\omega(n)$ denotes a positive increasing function such that $\omega(n) \to \infty$ as $n \to \infty$.

That the sequence $\theta, \theta^2, \ldots$ for almost all θ is uniformly distributed (modulo 1) had already been proved by KOKSMA [1]), whereas the sharpest estimate for the discrepancy of this sequence known till now was given by DREWES [2]).

§ 3. The theorems quoted above are contained in the following Theorem 3, which itself is a special case of the main Theorem 5.

Theorem 3. *Let $a<b$, $\delta>0$ be given real numbers. Let $f(1,\theta)$, $f(2,\theta)$, ..., denote a sequence of real functions which are defined on $a\leqq\theta\leqq b$, such that*

$$f'_\theta(n+1,\theta)\geqq(1+\delta)f'_\theta(n,\theta)>0;\quad f''_\theta(n+1,\theta)\geqq(1+\delta)f''_\theta(n,\theta)\geqq 0 \qquad (n=1,2,\ldots)$$

for all values of θ on $a\leqq\theta\leqq b$. Let $\omega(n)$ denote an increasing function of $n=1,2,\ldots$, such that $\omega(n)\to\infty$ as $n\to\infty$.

Then for almost all θ on $a\leqq\theta\leqq b$ the discrepancy in the uniform distribution of the sequence

$$f(1,\theta),\ f(2,\theta),\ldots$$

satisfies the relation (5).

Remark. It is clear that the sequences of the theorems 1 and 2 satisfy the conditions of Theorem 3. In the first case we put without loss of generality $a=0$, $b=1$ and in the second case we put $a=1+\delta$, $b>a$ and after application of Theorem 3, we let

$$\delta\to 0,\quad b\to\infty.$$

The reader will find the deduction of Theorem 3 from Theorem 4 in § 8.

§ 4. For the proof of our main theorem we deduce a lemma (Lemma 2), which has some interest in itself. For the special case, considered in Theorem 3, it runs as follows:

Theorem 4. *Suppose that the conditions of Theorem 3 are satisfied. Let K denote a positive constant. Then for almost all θ the following statement is true: If N and k are integer such that $1\leqq k\leqq N^K$, then*

$$\left|\sum_{n=1}^{N} e^{2\pi i k f(n,\theta)}\right|\leqq C(\theta)N^{\frac{1}{2}}\log^{\frac{1}{2}}N(\log\log N)^{\frac{1}{2}}\omega(N),$$

where $C(\theta)$ does not depend on N or k.

The reader finds its deduction in § 9.

§ 5. Before we state the main theorem, we make some

Preliminary Remarks. Let N and r denote positive integers. Out of the N integers $n=1,2,\ldots,N$, we can form N^r different r-tuples; such an r-tuple we shall denote by $(n_1,\ldots,n_r)$. There are C_N^r different r-tuples among them for which $n_1\leqq n_2\leqq\ldots\leqq n_r$. Such a special r-tuple we shall denote also by $\{n_1,\ldots,n_r\}$. The elements $n_1,\ldots,n_r$ of the r-tuple $\{n_1,\ldots,n_r\}$ have a number of different permutations, which we shall denote by $A\{n_1,\ldots,n_r\}$. Then we obviously have

$$A\{n_1,\ldots,n_r\}\leqq r! \qquad (6)$$

and

$$\sum_{\{n_1,\ldots,n_r\}} A\{n_1,\ldots,n_r\}=N^r. \qquad (7)$$

For later purposes we put

$$A_N^r = \sum_{\{n_1,\ldots,n_r\}} A^2\{n_1,\ldots,n_r\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Definition. If $\{n_1, \ldots, n_r\}$ and $\{m_1, \ldots, m_r\}$ are two different r-tuples of the above kind, we say that the first is greater than the second:

$$\{n_1,\ldots,n_r\} > \{m_1,\ldots,m_r\},$$

if and only if for some τ $(1 \leqq \tau \leqq r)$

$$n_\tau > m_\tau \quad , \quad n_\varrho = m_\varrho \quad (\varrho = \tau+1, \tau+2, \ldots, r).$$

Condition A. Let $g(x, \theta)$ for $x = 1, 2, \ldots, N$ denote a function of θ on the segment $a \leqq \theta \leqq b$ such that for each couple of r-tuples $\{n_1, \ldots, n_r\} > \{m_1, \ldots, m_r\}$ the function

$$\Phi(\theta) = \Phi(n_1,\ldots,n_r; m_1,\ldots,m_r; \theta) = \sum_{\varrho=1}^{r} g(n_\varrho, \theta) - \sum_{\varrho=1}^{r} g(m_\varrho, \theta) \quad . \quad (9)$$

has a derivative for $a \leqq \theta \leqq b$, which is continuous, $\neq 0$, and either non-decreasing or non-increasing in the segment $a \leqq \theta \leqq b$. We then put

$$\left.\begin{aligned} &\Psi = \Psi(n_1,\ldots,n_r; m_1,\ldots,m_r) = \\ &\text{Min}\,\{\Phi'_\theta(n_1,\ldots,n_r; m_1,\ldots,m_r; a), \Phi'_\theta(n_1,\ldots,n_r; m_1,\ldots,m_r; b)\}, \end{aligned}\right\} \quad . \quad (10)$$

$$\left.\begin{aligned} B_N = N^{-r} \sum_{\{n_1,\ldots,n_r\} > \{m_1,\ldots,m_r\}} \sum A\{n_1,\ldots,n_r\}\, A\{m_1,\ldots,m_r\} \\ \Psi^{-1}(n_1,\ldots,n_r; m_1,\ldots,m_r). \end{aligned}\right\} \quad (11)$$

Now we state our main theorem:

Theorem 5. I. *Let a and b denote real constants with $a < b$. Let $f(n, \theta)$ for $n = 1, 2 \ldots$ denote a real function of θ on the segment $a \leqq \theta \leqq b$. Let N_0 be a positive integer. Let $r = r(N)$ and $s = s(N)$ be positive integers which are defined for each integer $N \geqq N_0$, such that*

$$S(N) \leqq N.$$

Let for each integer $N \geqq N_0$ and each integer $\sigma = 1, \ldots, s(N)$ the N_σ functions

$$g_\sigma(x, \theta) = f(\sigma + (x-1)\, s, \theta) \qquad \left(x = 1, 2, \ldots, N_\sigma = \left[\frac{N-\sigma}{s}\right] + 1\right)$$

be considered and let the condition A be satisfied with g_σ instead of g and with N_σ instead of N.

II. *Putting*

$$B_N^* = \max_{1 \leqq \sigma \leqq s} B_{N_\sigma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

we assume that a non-decreasing sequence $\psi(1), \psi(2), \ldots$ of positive numbers exists such that the series

$$\sum_{n=N_0}^{\infty} s \cdot \{(b-a)\, r!\, N^{\frac{1}{2}} + B_N^* \log N\} \left\{\psi\left(\left[\frac{N-s}{s}\right] + 1\right)\right\}^{-2r} \quad . \quad (13)$$

converges. Then almost all numbers θ of $a \leqq \theta \leqq b$, have the property that the discrepancy $D(N)$ of the sequence $f(1,\theta), f(2,\theta), \ldots$ satisfies the inequality

$$ND(N) \leqq K_1 s^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \cdot \psi\left(\left[\frac{N-1}{s}\right]+1\right) \log N \text{ for } N \geqq N_0^*, \quad (14)$$

where K_1 denotes a numerical constant, whereas N_0 denotes an index depending on θ.

§ 6. We now prove

Lemma 1. *Let N and r denote positive integers and let $g(x,\theta)$ satisfy the condition A. Then for each fixed integer $h \neq 0$ we have*

$$\int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} d\theta = (b-a) A_N^r + \frac{2\vartheta}{\pi h} B_N N^r \quad (|\vartheta| \leqq 1). \quad (15)$$

Proof.

$$\left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} = \left\{ \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right\}^r \left\{ \sum_{x=1}^{N} e^{-2\pi i h g(x,\theta)} \right\}^r$$

$$= \sum_{(n_1,\ldots,n_r)} e^{2\pi i h (g(n_1,\theta)+\ldots+g(n_r,\theta))} \cdot \sum_{(n_1,\ldots,n_r)} e^{-2\pi i h (g(n_1,\theta)+\ldots+g(n_r,\theta))},$$

where both sums are to be expanded over all N^r r-tuples of integers

$$n_\varrho = 1, 2, \ldots, N.$$

Applying the preliminary remark of § 5, we write

$$\left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} = \sum_{|n_1,\ldots,n_r|} A\{n_1,\ldots,n_r\} e^{2\pi i h (g(n_1,\theta)+\ldots+g(n_r,\theta))}$$

$$\cdot \sum_{|n_1,\ldots,n_r|} A\{n_1,\ldots,n_r\} e^{-2\pi i h (g(n_1,\theta)+\ldots+g(n_r,\theta))}$$

$$= \sum_{|n_1,\ldots,n_r|} \sum_{|m_1,\ldots,m_r|} A\{n_1,\ldots,n_r\} A\{m_1,\ldots,m_r\} e^{2\pi i h \left\{ \sum_{\varrho=1}^{r} g(n_\varrho,\theta) - \sum_{\varrho=1}^{r} g(m_\varrho,\theta) \right\}}$$

$$= \sum_{|n_1,\ldots,n_r|} A^2\{n_1,\ldots,n_r\} + 2 \sum_{|n_1,\ldots n_r| > |m_1,\ldots,m_r|} \sum A\{n_1,\ldots,n_r\} A\{m_1,\ldots,m_r\} \cos 2\pi h \Phi(\theta)$$

by (9). Hence by (8)

$$\left.\begin{aligned} \int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\theta)} \right|^{2r} d\theta &= (b-a) A_N^r + \\ + 2 \sum_{|n_1,\ldots,n_r| > |m_1,\ldots,m_r|} \sum A\{n_1,\ldots,n_r\} A\{m_1,\ldots,m_r\} &\int_a^b \cos 2\pi h \Phi(\theta)\, d\theta. \end{aligned}\right\} \quad (16)$$

Now choosing the new variable of integration u by the substitution

$$u = \Phi(\theta),$$

we find

$$du = \Phi'(\theta)\, d\theta$$

and

$$\int_a^b \cos 2\pi h\, \Phi(\theta) \cdot d\theta = \int_{\Phi(a)}^{\Phi(b)} \cos 2\pi h u\, \frac{du}{\Phi'(\theta)}$$

and therefore using BONNET's theorem and (10)

$$\left| \int_a^b \cos 2\pi h\, \Phi(\theta) \cdot d\theta \right| \leqq \frac{1}{\pi h\, \Psi}.$$

Hence by (16) and (11)

$$\int_a^b \left| \sum_{x=1}^{N} e^{2\pi i h g(x,\vartheta)} \right|^{2r} d\theta = (b-a)\, A_N^r + \frac{2\vartheta N^r}{\pi h} B_N \cdot (|\vartheta| \leqq 1).$$

Q.e.d.

Lemma 2. *Let the conditions* I *of the main theorem* 5 *be satisfied.* II. *Defining* B_N^* *by* (12), *we suppose that a non-decreasing sequence* $\psi(1), \psi(2), \ldots$ *and a non-decreasing sequence of integers* $\Lambda(1), \Lambda((2), \ldots$ $(\Lambda(N) \geqq 3,$ *if* $N \geqq N_0)$ *exist, such that the series*

$$\sum_{n=N_0}^{\infty} s\{(b-a)\, r / \Lambda(N) + 2 B_N^* \log \Lambda(N)\} \left\{ \psi\left(\left[\frac{N-s}{s} \right] + 1 \right) \right\}^{-2r}. \quad (13a)$$

converges. Then almost all numbers θ *of* $a \leqq \theta \leqq b$ *have the property that for all integers* $h = 1, 2, \ldots, \Lambda(N)$

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n,\vartheta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s} \right] + 1 \right), \text{ if } N \geqq N_0^*(\theta) \ . \quad (17)$$

Proof. Let N be a fixed integer $\geqq N_0$. Let (h, σ) denote a couple of integers which satisfy the inequalities

$$1 \leqq h \leqq \Lambda(N), \qquad 1 \leqq \sigma \leqq s.$$

Then the Lemma 1 with

$$g(x, \theta) = g_\sigma(x, \theta) = f(\sigma + (x-1)\, s, \theta), \quad N = N_\sigma$$

learns

$$\int_a^b \left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x,\vartheta)} \right|^{2r} d\theta \leqq (b-a)\, r / N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r \quad . \quad . \quad (18)$$

because of (6) and (7).

Now let $S(N, h, \sigma)$ denote the set of all numbers θ on $a \leqq \theta \leqq b$ for which

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x,\theta)} \right| \geqq N_\sigma^{\frac{1}{2}} \psi(N_\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then by (18) we obviously find for its measure $mS(N, h, \sigma)$ the inequality

$$m S(N, h, \sigma) N_\sigma^r \{\psi(N_\sigma)\}^{2r} \leqq (b-a) r! N_\sigma^r + \frac{1}{h} B_{N_\sigma} N_\sigma^r,$$

hence

$$m S(N, h, \sigma) \leqq \left\{(b-a) r! + \frac{1}{h} B_{N_\sigma}\right\} \{\psi(N_\sigma)\}^{-2r}$$

and therefore by

$$N_\sigma \geqq \left[\frac{N-s}{s}\right] + 1 \text{ and (12):}$$

$$m S(N, h, \sigma) \leqq \left\{(b-a) r! + \frac{1}{h} B_N^*\right\} \left\{\psi\left(\left[\frac{N-s}{s}\right] + 1\right)\right\}^{-2r}.$$

Now let $S(N)$ denote the set

$$S(N) = \sum_{(h,\sigma)} S(N, h, \sigma),$$

where the summation is to be extended over all couples (h, σ) which satisfy $1 \leqq h \leqq \Lambda(N)$, $1 \leqq \sigma \leqq s$. Then we have

$$\left.\begin{aligned} mS(N) &\leqq \left\{(b-a) r! s \Lambda(N) + s B_N^* \sum_{h=1}^{\Lambda(N)} \frac{1}{h}\right\} \left\{\psi\left(\left[\frac{N-s}{s}\right]+1\right\}^{-2r}\right. \\ &< s\{(b-a) r! \Lambda(N) + 2 B_N^* \log \Lambda(N)\} \left\{\psi\left(\left[\frac{N-s}{s}\right]+1\right)\right\}^{-2r}. \end{aligned}\right\} \quad (20)$$

Each θ of $a \leqq \theta \leqq b$, which does not belong to $S(N)$ $(N \geqq N_0)$ has the property that the inequality

$$\left| \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x,\theta)} \right| \leqq N_\sigma^{\frac{1}{2}} \psi(N_\sigma) < \left(\frac{N-1}{s} + 1\right)^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right] + 1\right)$$

is valid for all couples (h, σ) which satisfy the inequalities

$$1 \leqq h \leqq \Lambda(N) \ , \ 1 \leqq \sigma \leqq s.$$

Therefore we have for such a θ

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \leqq \left| \sum_{\sigma=1}^{s} \sum_{x=1}^{N_\sigma} e^{2\pi i h g_\sigma(x,\theta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right] + 1\right),$$

for all integers $h = 1, 2, \ldots, \Lambda(N)$.

Now as $mS(N)$ satisfies (20) and as the series (13a) converges, almost all numbers θ of $a \leqq \theta \leqq b$ belong to at most a finite number of the sets

$S(N)$ $(N = N_0, N_0 + 1, \ldots)$. Therefore for almost all θ of $a \leqq \theta \leqq b$ an index N_0^* can be found, such that

$$\left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \leqq 2 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right]+1\right),$$

whatever the value of the integer $h = 1, 2, \ldots \Lambda(N)$ may be. Q.e.d.

§ 7. In order to prove the main theorem, we quote the following theorem, which is an improvement proved by ERDÖS–TURÁN [3]) of the one dimensional case of a theorem of VAN DER CORPUT–KOKSMA [1]).

Lemma 3. *If $u_1, u_2, \ldots$ is a real sequence and if $D(N)$ denotes its discrepancy, then for each integer $m \geqq 1$, we have*

$$ND(N) \leqq K\left\{\frac{N}{m+1} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h u_n} \right|\right\}, \quad . \quad . \quad . \quad (21)$$

where K denotes a numerical constant.

Proof of the Theorem 5. Put

$$u_n = f(n, \theta), \; m = [\sqrt{N}], \; \Lambda(N) = [\sqrt{N}].$$

Then (using Lemma 2) for almost all θ we have by (17) and (21), if $N \geqq N_0^*(\theta)$

$$ND(N) \leqq K\sqrt{N} + K \sum_{h=1}^{[\sqrt{N}]} \frac{2}{h} s^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right]+1\right)$$

$$\leqq K_1 s^{\frac{1}{2}} N^{\frac{1}{2}} \psi\left(\left[\frac{N-1}{s}\right]+1\right) \log N.$$

Q.e.d.

§ 8. **Proof of Theorem 3.** Be $\omega(N)$ the function of Theorem 3 and let N_0 be a sufficiently large integer. We shall prove that the functions $f(n, \theta)$ of Theorem 3 satisfy the conditions of Theorem 5, if we put for $N \geqq N_0$

$$\left.\begin{aligned} s(N) &= \left[\frac{2}{\log(1+\delta)} \log\log N\right]; \\ r(N) &= \left[\frac{\log N}{\log \sqrt{\omega([\sqrt{N}])}}\right] + 1; \; \psi(N) = \sqrt{\log N^2} \cdot \sqrt{\omega(N)} \end{aligned}\right\} \quad . \quad (22)$$

where δ denotes the constant of Theorem 3. Now for $N \geqq N_0$ we consider the $s = s(N)$ sequences

$$g_\sigma(x, \theta) = f(\sigma + (x-1)s, \theta) \left(1 \leqq \sigma \leqq s; \; x = 1, 2, \ldots, N_\sigma = \left[\frac{N-\sigma}{s}\right] + 1\right).$$

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[3]) P. ERDÖS and P. TURÁN, On a problem in the theory of uniform distribution. Proc Kon. Ned. Akad. v. Wetensch., Amsterdam, **51**, 1146—1154, 1262—1269 (1948); Ind. Math. **10**, 370—378, 406—413 (1948).

By hypothesis we have (the prime meaning differentiation by θ):

$$\frac{f'(n+1,\theta)}{f'(n,\theta)} \geqq 1+\delta,$$

hence for $N \geqq N_0$

$$\frac{g'_\sigma(x+1,\theta)}{g'_\sigma(x,\theta)} \geqq (1+\delta)^s > (1+\delta)^{3/2 \frac{\log\log N}{\log(1+\delta)}} = (\log N)^{3/2} > r+1.$$

Therefore if we take two r-tuples

$$\{n_1,\ldots,n_r\} > \{m_1,\ldots,m_r\},$$

we have, using the notation of the Preliminary Remarks:

$$\begin{aligned}\Phi'(\theta) &= g'_\sigma(n_1,\theta)+\ldots+g'_\sigma(n_r,\theta)-g'_\sigma(m_1,\theta)-\ldots-g'_\sigma(m_r,\theta)\\ &\geqq g'_\sigma(n_\tau,\theta)-r\,g'_\sigma(n_\tau-1,\theta) > g'_\sigma(n_\tau-1,\theta) \geqq f'(1,a) = c_0 \quad (23)\end{aligned}$$

and we conclude that if we range the r-tuples $\{n_1,\ldots,n_r\}$ in order of increasing magnitude, the corresponding sums

$$g'_\sigma(n_1,\theta)+\ldots+g'_\sigma(n_2,\theta)$$

with each step will increase by at least the amount c_0, whatever the value of θ $(a \leqq \theta \leqq b)$ may be. Hence, if the r-tuple $\{n_1,\ldots,n_r\}$ is fixed, we have by (10) and (9)

$$\sum_{|n_1,\ldots,n_r|>|m_1,\ldots,m_r|} \sum \{\Psi(n_1,\ldots n_r;m_1,\ldots,m_r\}^{-1} \leqq \frac{1}{c_0}\sum_{k=1}^{N_\sigma^r}\frac{1}{k} \leqq \frac{1+r\log N}{c_0},$$

as there are at most $N_\sigma^r \leqq N^r$ r-tuples $\{m_1,\ldots,m_r\}$.

Therefore we find by (11) and (6)

$$B_{N_\sigma} \leqq N_\sigma^{-r}\, r!\,\frac{1+r\log N}{c_0}\sum_{|n_1,\ldots,n_r|} A\{n_1,\ldots,n_r\} = \frac{r!}{c_0}(1+r\log N)$$

by (7). Hence we find by (12) a fortiori

$$B_N^* \leqq \frac{2r}{c_0}\, r!\log N < r^r \log N \text{ for } N \geqq N_0.$$

Thus we find that the general term of our series (13) is at most

$$\{(b-a)s r^r N^{\frac{1}{2}} + s r^r \log^2 N\}\{\psi([\sqrt{N}]+1)\}^{-2r} \leqq$$

$$\leqq c_1 s (\log N)^r N^{\frac{1}{2}} \cdot \sqrt{\log N}^{-2r}\{\sqrt{\omega([\sqrt{N}])}\}^{-2r}$$

by (22) and therefore by (22)

$$\leqq c_1 s N^{\frac{1}{2}}\{\sqrt{\omega([\sqrt{N}])}\}^{-\frac{2\log N}{\log\sqrt{\omega([\sqrt{N}])}}} < N^{-\frac{5}{4}}, \text{ if } N \geqq N_0.$$

Hence, our series (13) converges.

By hypothesis we further have $f''_{(n+1,\theta)} \geqq (1+\delta)\, f''_{(n,\theta)} \geqq 0$.

Repeating the proof of (23) with f'' instead of f' and g''_σ instead of g'_σ, we find $\Phi''(\theta) \geqq 0$. Hence $\Phi'(\theta)$ is non-decreasing.

From our result we conclude that for almost all θ the inequality (14) holds, i.e. because of (22):

$$ND(N) \leqq K_2 N^{\frac{1}{2}} (\log N)^{\frac{1}{2}} (\log\log N)^{\frac{1}{2}} \sqrt{\omega(N)}, \text{ if } N \geqq N_0^*(\theta).$$

Hence (5) follows immediately.

§ 9. **Proof of Theorem 4.** We shall use Lemma 2 and we put

$$\left.\begin{aligned} A(N) = [N^K],\ s(N) = \left[\frac{2}{\log(1+\delta)} \log\log N\right],\ r(N) = \\ = \left[\frac{(K+2)\log N}{\log \omega([\sqrt{N}])}\right] + 1,\ \psi(N) = \sqrt{\log N^2}\ \ \omega(N), \end{aligned}\right\} \quad (24)$$

where K, δ and $\omega(N)$ are defined in Theorem 4. Then in exactly the same way as in § 8 it follows that

$$B_N^* \leqq r^r \log N \text{ for } N \geqq N_0$$

and thus the general term of the series (13a) is at most

$$\{(b-a)\, s r^r N^K + s r^r \cdot 2K \log^2 N\}\, \{\psi([\sqrt{N}]+1)\}^{-2r} \leqq$$

$$\leqq c_2\, s (\log N)^r N^K (\sqrt{\log N})^{-2r}\, \{\omega([\sqrt{N}])\}^{-2r}$$

by (24) and therefore by (24)

$$< c_2 s \cdot N^K \{\omega([\sqrt{N}])\}^{\frac{-(2+K)\log N}{\log \omega([\sqrt{N}])}} < N^{-\frac{1}{2}}, \text{ if } N \geqq N_0.$$

Hence, the series (13a) converges. From our result we conclude that (17) holds for almost all θ on $a \leq \theta \leq b$, i.e.

$$\left| \sum_{n=1}^{N}{}' e^{2\pi i h f(n,\theta)} \right| \leqq K_3 N^{\frac{1}{2}} \log^{\frac{1}{2}} N (\log\log N)^{\frac{1}{2}} \omega(N), \text{ if } N \geqq N_0^*(\theta).$$

From this Theorem 4 follows immediately.

Botany. — *De F_4-zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris.* III. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of October 30, 1948.)

Cl. 8, Form. l b th. Alle 3 afmetingen zijn klein. 30 gevallen. Bij de bonen van cl 8, met de form. l b th, kunnen we 3 vormen onderscheiden. Van bonen van cl 8a komen de indices overeen met die van bonen van cl 2, met de form. L B th als van de I-lijn, — de dikte van deze bonen is relatief klein —. De formule is l b th I. Van bonen van cl 8b komen de indices overeen met die van bonen van cl 7, met de form. l b Th als van de II-lijn, de dikte van deze bonen is relatief groot. De formule is l b th II. In onderclasse 8c brengen we de bonen met de form. l b th van de uitgangsbonen, waarbij alle 3 afmetingen gelijkmatig klein zijn. Deze laatste uitgangsbonen zijn, genotypisch, uitsplitsingen met de 3 afmetingen als geëxtraheerde recessieven (ll bb thth). Het zijn bonen met de form. l b th in engere zin. De L B-index van bonen van cl 8c is L B = 100 × b/l, komt overeen met die van bonen van de II-lijn en is dus L B = 70. De L Th-index (= 100 × th/l) en de B Th-index (= 100 × th/b) zijn intermediair ten opzichte van deze indices van de I- en de II-lijn. Deze verdeling in 3 onderclassen hebben we in tab. 1 en 1a nog niet doorgevoerd.

In 17 gevallen is de formule van de gemiddelden der bonenopbrengsten ook l b th.

Van pl. 270 (tab. 1 en 1a) is de uitgangsboon van pl. 62. De formule van de gemiddelden van pl. 62 is L B Th (l_m = 13.8 mm). De uitgangsboon voor pl. 62 is van pl. 70, F_2-1934, de formule is L B Th. De formule van de gemiddelden van pl. 70 is eveneens L B Th. De 5 bonen van de peul van de uitgangsboon van pl. 62 voor pl. 270 zijn nog al verschillend. Drie bonen hebben grotere afmetingen, vooral ook een grotere breedte. Bij de bonenopbrengst van pl. 270 staat aangetekend „slecht, onrijp?". Deze bonen kunnen onvolgroeid en daardoor klein zijn. Het gewicht van de bonen is nog al verschillend (van de verschillende peulen: 31—38, 26—31, 21—30, 24—28 cg, en 40, 48—52 cg). Alleen deze laatste peul met 4 bonen heeft 2 bonen met een grote lengte (l = 13.7 en = 13.6 mm). Alle bonen op 2 na behoren tot cl 8a (2 tot cl 4, tab. 1a). Alleen verder kweken zou hebben kunnen leren, of we in dit geval met erfelijkheid van de kleine afmetingen te doen hebben.

Pl. 341 (tab. 1 en 1a) en pl. 342 (zie ook pl. 340, blz. 275) zijn van uitgangsbonen van pl. 89. De uitgangsboon voor pl. 341 is de 2de van de 4 bonen van de peul, die alle kleine afmetingen hebben. Alle bonen van pl. 89 zijn klein; er zijn slechts 4 bonen met een iets grotere lengte (l = 13.8—12.9 mm). De gemiddelde lengte van pl. 341 is l_m = 12.0 mm. Een dergelijke gemiddelde lengte treffen we ook bij bonenopbrengsten van de II-lijn van 1936 aan (tab. 4b). Bij de bonenopbrengst van pl. 341 staat aangetekend „heel goed, van de bonen echter de meeste gevlekt (en

rimpelig)". De grootste lengte van de bonen is l = 14.4 mm, dan volgt l = 12.9 mm. Deze ene boon met een grote lengte is opmerkelijk, want deze lengte ligt buiten de variatiebreedte van de lengte van de bonen van de II-lijn, dus met de formule $l_1 l_2$. We moeten aannemen, dat de formule van de lengte van de uitgangsboon ook één of meer L-factoren bevat (zie ben.). Er is in de bonenopbrengst van 27 bonen maar één boon met een grote lengte. Het is moeielijk, om het verschijnen van deze ene boon met een grote lengte door uitsplitsing te verklaren. Wellicht zijn er onder de 6 bonen met een niet zeer kleine lengte (l = 12.5—12.9 mm) enkele bonen met een L L- of L l-verbinding in de formule van de lengte. De variatiebreedte van bonen van de I-lijn is zeer groot. Volgens de classificatie van de bonenopbrengst van pl. 341 (tab. 1a) zijn er 41 bonen in cl 8. Hiervan behoren er 13 tot cl 8c. De formule van de uitgangsboon van pl. 89 voor pl. 341 is l b th in bijna homozygote vorm. Ze heeft slechts enkele lengte- en breedtefactoren als L l en B b. Van de bonenopbrengst van pl. 342 is de grootste lengte der bonen l = 13.3 mm (dan volgt l = 12.9 mm). Ook hier treffen we dus een boon aan met een grotere lengte dan bij de II-lijn voorkomt. Bovendien zijn er te veel bonen met een nog al grote lengte. Volgens de classificatie zijn er 19 bonen in cl 8, waarvan 3 in cl 8a, 12 in cl 8c en 4 in cl 8b. Bovendien zijn er 4 bonen in cl 7 en één in cl 1. De uitgangsboon van pl. 89 voor pl. 342 verschilt in haar genotype iets van die voor pl. 341. Ze is l b th en bevat ook enkele Th Th-verbindingen. Van de slechte bonenopbrengst van pl. 340 behoren alle 14 bonen tot cl 8; 8 tot cl 8c en 6 tot cl 8a. Er is overeenkomst met pl. 341.

Van pl. 343 is de uitgangsboon van pl. 90; haar lengte is nog al groot (l = 12.8 mm). Er is in de ascendentie een vrij grote lengte en een kleine breedte. De bonenopbrengst van pl. 343 bevat 2 bonen met een grote lengte (l = 15.1 en = 14.7 mm). Van alle bonen is de breedte klein (b = 8.5—7.1 mm); ook de dikte (th = 6.2—5.1 mm). Volgens de classificatie zijn er 10 bonen in cl 4. form. L b th en 16 bonen in cl 8. De uitgangsboon van pl. 90 voor pl. 343 heeft de form. l b th met enkele lengtefactoren als L L en L l en de overige als l l.

De uitgangsboon voor pl. 351 is van pl. 92; ze heeft zeer kleine afmetingen. De grootste lengte van de bonen van pl. 351 is l = 14.1 mm, (2 bonen, dan volgt l = 13.1 mm). Bij de uitgangsboon, die een zeer kleine lengte heeft (l = 10.7 mm) moeten we dus aannemen, dat ook L-factoren in de formule aanwezig zijn. In ons materiaal van de I-lijn van 1936 zijn ook bonen met een zeer kleine lengte (l = 10.2 en 11.2 mm). Volgens de classificatie van pl. 351 zijn er zeer veel bonen in cl 8 en een enkele in cl 1, 2 en 6. Onder de bonen van cl 8 zijn er verschillende, die behoren tot de onderclasse 8c. Tot overeenkomstige resultaten voert het onderzoek van 2 andere gevallen (pl. 365 en 368). Ook hier bevatten de bonenopbrengsten goede voorbeelden van bonen van cl 8c. Van een ander geval tenslotte nog (pl. 402) is de uitgangsboon iets meer heterozygoot voor de form. l b th. Volgens de classificatie zijn hier ook bonen in cl 7, 5 en 4.

Van de overige gevallen, waarin de uitgangsboon de formule l b th, cl 8 heeft en ook de formule van de gemiddelden der bonenopbrengsten de formule l b th cl 8 is, zijn er 8, waar de uitgangsbonen van dezelfde plant (pl. 51, F_3-1935, fig. 5) genomen zijn.

Van pl. 51, F_3-1935 zijn in 1936 de F_4-bonenopbrengsten van 35 uitgangsbonen onder-

zocht (pl. 977—1011 en 224). Voor pl. 51 is de uitgangsboon van pl. 66, F_2-1934 (tab. 21, 1947) genomen. De formule is L B th, cl 2 (th = 6.5 mm); de formule van de gemiddelden van pl. 51 is l b th (l_m = 12.9 mm). Volgens de classificatie zijn er in de bonenopbrengsten van 60 bonen van pl. 51, 18 bonen in cl 7. Van de 35 uitgangsbonen, die van pl. 51 genomen zijn, behoren er 17 tot cl 7, 8 tot cl 8 en 10 tot de overige classen (de grootste lengte is l = 14.2 en = 13.8 mm). Er zijn daarbij relatief zeer veel bonen van cl 7. De bonenopbrengsten van de pl. 977—1011 en 224 behoren alle tot cl 8. We nemen aan, dat aan de grootte-verschillen van de bonen van de bonenopbrengsten van F_4-1936 en F_3-1935 ook milieu-invloeden aandeel hebben. De bonen van 1936 zijn gemiddeld iets kleiner dan die van 1935.

Het blijkt, dat bonenopbrengsten met de form. l b th der gemiddelden in haar samenstelling overeenkomst hebben met bonenopbrengsten met de form. L B Th. In beide gevallen is er meestal een grote mate van heterozygotie.

De gemiddelde lengten van de 35 bonenopbrengsten variëren van 11.0—13.0 mm, die van 25 bonenopbrengsten van de II-lijn van 1936 variëren van 10.1—12.0 mm. De formule van de II-lijn is ll bb Th Th. Van de bonenopbrengsten met een groter gemiddelde lengte dan 12.0 mm, moet de formule van de lengte van de uitgangsboon dus enkele L l- of L L-verbindingen bevatten. Deze bonenopbrengsten bevatten meestal enkele bonen met een lengte, die groter is dan 13.0 mm, (bonen met een grotere lengte komen onder de bonen van de II-lijn van 1936 niet voor, zie tab. 8). Ook bij gevallen, waar de uitgangsbonen een andere formule hebben dan l b th (d.w.z., met L in de formule, terwijl de formule van de gemiddelden l b th is, nemen we aan, dat de uitgangsboon L l- of L L-verbindingen bevat. Van deze verschillende mogelijkheden vinden we voorbeelden bij de 35 gevallen. Volgens de classificatie bevatten de bonenopbrengsten in deze gevallen bijna alleen bonen in cl 8, resp., in cl 8a, cl 8b, cl 8a en 8c, en cl 8a en 8b.

Van de 17 van de 35 bonenopbrengsten met uitgangsbonen van pl. 51, waar deze de form. l b Th hebben, bevatten, volgens de classificatie der bonenopbrengsten, er enige zeer veel bonen in cl 8b en ook daarbij soms enige in cl 7. De uitgangsbonen hebben in deze gevallen de form. l b Th in meer of minder homozygote vorm. In andere gevallen zijn de indices intermediair met bonen in cl 8 en de meeste andere classen.

Van de 8 van de 35 gevallen, waar de formule van de uitgangsboon l b th is en ook die van de gemiddelden van de bonenopbrengsten (z. boven), kunnen we ook, volgens de classificatie, de bonenopbrengsten onderscheiden naar gelang de uitgangsbonen de formules l b th I, l b th II of l b th hebben. Goede voorbeelden van deze bonenopbrengsten, dus met overwegend de eigenschappen van de bonen van cl. 8a, 8b en 8c zijn de pl. 978, 981 en 1000 (tab. 1 en 1a).

Van de uitgangsboon van pl. 51 voor pl. 978 hebben de afmetingen de grenswaarden, de indices zijn intermediair. De bonenopbrengst van pl. 978 is „goed". Er zijn te veel bonen met een niet zeer kleine lengte (5 bonen met l = 12.7—12.3 mm). De formule van de lengte van de uitgangsboon bevat dus een enkele L l of L L-verbinding. Er zijn 10 bonen met de L B-index 59—63, 11 met de L Th-index 46—50 en 13 met de B Th-index 75—78. De bonen van pl. 978 zijn overwegend intermediair, hebben slechts enkele bonen in cl 8b, de overige in cl 8a. De uitgangsboon voor pl. 1000 komt zeer met die voor pl. 978 overeen, ook de gemiddelden van de bonenopbrengst. De gemiddelde L B-index is iets hoger. Er zijn, volgens de classificatie, 24 bonen in cl 8a, waarvan 16 in cl 8c. De samenstelling van de bonenopbrengst is iets eenvormiger dan die van pl. 978, beantwoordt iets meer aan die, waarvan de formule van de uitgangsboon l b th in engere zin is. Pl. 981 verschilt duidelijk van pl. 978 en 1000, komt overeen met bonenopbrengsten van de II-lijn. Van de uitgangsboon van pl. 51 naderen de indices zeer tot die van bonen van de II-lijn (tab. 7). Gemiddelden als van pl. 1000 treffen we onder de bonenopbrengsten van de II-lijn van 1936 niet aan (tab. 4b). Ook van de individuele bonen van pl. 981 naderen de indices tot die van bonen van de II-lijn van 1936 (tab. 7), maar komen er niet geheel mee overeen.

Het onderzoek van de bonenopbrengst van pl. 51 F_3-1935 voerde tot het resultaat, dat haar uitgangsboon van pl. 66, F_2-1934 de formule 1 b Th heeft met veel heterozygotie (1948). Uit het onderzoek van de 35 bonenopbrengsten van de pl. 977-1011 en 224, F_4-1936, waarvan de uitgangsbonen van pl. 51 genomen zijn, blijkt, dat deze uitgangsbonen genotypische verschillen bezitten, overwegend binnen het gebied van bonen met de formule 1 b th en dat geen der uitgangsbonen de formule 1 b th, cl 8a, 8b en c of de form. 1 b Th, cl 7 in geheel homozygote vorm hebben.

Van de gevallen, waar de uitgangsboon de formule 1 b th, cl 8 heeft en ook de formule van de gemiddelden 1 b th is, zijn er nu nog 4 te vermelden (pl. 365, 368, 1048 en 402). Alleen van pl. 1048 (tab. 1 en 1a) volgt hier een korte beschrijving.

De uitgangsboon is van pl. 101, F_3-1935; ze is de voorlaatste boon uit de peul en heeft veel kleinere afmetingen dan de 4 voorafgaande bonen van de peul. Bonen met deze afmetingen en indices komen onder de bonen van de II-lijn en de I-lijn van 1935 niet voor. Dit is opmerkelijk. Indien we met een erfelijke variatie te doen hebben, is ze een voorbeeld van transgressieve variabiliteit (Proc. 52, blz. 80). Volgens de classificatie (tab. 1a) zijn bijna alle bonen in cl 8a, form. 1 b th I. De gemiddelde indices naderen zeer tot die van bonenopbrengsten van de I-lijn. Van de individuele bonen komen enkele geheel met bonen van de I-lijn overeen. De uitgangsboon voor pl. 1049 (tab. 1 en 1a) is de laatste boon uit dezelfde peul, waarvan de uitgangsboon voor pl. 1048 genomen is. Ze is zeer klein. Waarschijnlijk hebben we bij deze laatste boon van de peul en eveneens bij de voorlaatste voor pl. 1048, met modificaties te doen van bonen met de form. 1 b th.

Van pl. 101, F_3-1935 zijn in 1936 de bonenopbrengsten van 19 uitgangsbonen onderzocht (pl. 1019—1049 en 390). De uitgangsbonen behoren tot verschillende classen en de meeste bonenopbrengsten vonden reeds bespreking (cl 1b, 2b, 4, 5). Volgens de classificatie zijn er in de bonenopbrengst van 83 bonen van pl. 101, 36 bonen in cl 1 en de overige bonen zijn over alle classen verspreid (behalve cl 6). Bij de uitzaai van F_4-1936, staat bij de uitzaaibonen van pl. 101 aangetekend „fraaie F_3-generatie, zelf".

Voor pl. 101 is de uitgangsboon van pl. 83, F_2-1934 genomen. Van 12 planten (pl. 97—109) zijn de uitgangsbonen van pl. 83 (tab. 21, 1948). De formule van de gemiddelden van pl. 83 is L B Th. De gemiddelde afmetingen zijn nog al groot (l_m = 14.2 mm). De formule van de gemiddelden van pl. 101 is ook L B Th (l_m = 13.5 mm). De uitgangsbonen van pl. 101 voor de pl. 1019—1049 en 390 zijn nog al verschillend (de grootste lengten zijn l = 15.1 en = 15.0 mm, dan volgen 14.7—12.1 mm, en de kleinste lengten zijn l = 10.9 en = 8.6 mm).

Van de 19 uitgangsbonen van de bonenopbrengst van 83 bonen van pl. 101, behoren er 9 tot cl 1, 1 tot cl 3, 2 tot cl 4, 1 tot cl 5, 1 tot cl 7 en 5 tot cl 8. Ook de gemiddelden van de bonenopbrengsten verschillen zeer; l_m = 14.2—12.0 mm. Volgens de classificatie zijn er in geen van deze 19 bonenopbrengsten bonen in cl 7 en cl 5. Hieruit blijkt, dat al de uitgangsbonen van deze 19 bonenopbrengsten het genotype van de bonen van het gebied van de I-lijn hebben, d.i., van cl 2 en cl 4, ook van cl 1 en cl 8. Deze 19 bonenopbrengsten van F_4-1936 geven dus een aanvulling van onze gegevens van de bonenopbrengst van pl. 101, F_3-1935 en haar classificatie.

Van 8 gevallen is de formule van de uitgangsboon 1 b th, cl. 8 en hebben de gemiddelden van de bonenopbrengsten een andere formule. We bespraken ze reeds (blz. 4, 9, 12).

In 24 gevallen is de formule van de gemiddelden 1 b th, cl 8 en heeft de uitgangsboon een andere formule. Er zijn uitgangsbonen uit alle classen. behalve uit cl 6.

Van de pl. 279 en 280 hebben de uitgangsbonen de form. $L_1 L_2$ B Th, cl 1a. Ook van de pl. 275, 277 en 278 alle van dezelfde pl. 63, F_3-1935, hebben de uitgangsbonen grote afmetingen en de bonenopbrengsten hebben kleine of vrij kleine gemiddelde afmetingen. De ascendentie geeft geen duidelijke aanwijzing voor de verschillen. Waarschijnlijk hebben we hier vooral te doen met modificatie. In 1935 bevatten de bonenopbrengsten vaak grote bonen, in 1936 waren de bonen vaak kleiner. Meermalen is aangetekend: „oogst slecht, enkele peulen nog groen, onrijp, waarschijnlijk te vroeg geoogst". Er is een grote niet-erfelijke variabiliteit. Toch is er verschil met de niet-erfelijke variabiliteit van de zuivere lijnen. We vinden voor de bonenopbrengsten van de I-lijn van 1936 na uitgangsbonen van 1935 (zie fig. 1, 1947, l.c.) b.v., van de uitgangsboon van 1935, $l = 21.6$ mm, en van de gemiddelden van de bonenopbrengst van 1936, $l_m = 14.5$ mm; evenzo $l = 19.6$ mm, $l_m = 13.4$ mm; ook $l = 15.5$ mm, $l_m = 12.9$ mm; ook $l = 11.2$ mm, $l_m = 13.8$ mm. De uitgangsbonen van planten van 1935 met de allergrootste lengten leveren bonenopbrengsten in 1936 met dezelfde gemiddelden als de uitgangsbonen van 1935 met de kleinste lengten. (Er is geen selectie.) Hier, bij ons F_4-materiaal van 1936, zien we, dat uitgangsbonen F_3-1935 met een zeer grote lengte, bonenopbrengsten van F_4-1936 hebben met zeer kleine gemiddelden. De erfelijke samenstelling van de uitgangsbonen heeft hier ook een aandeel in het resultaat, nemen we aan.

Onzeker in hunne betekenis zijn ten dele ook de overige 22 gevallen. De uitgangsbonen hebben de formule van alle classen, behalve van cl 6, form. 1 B th. In meerdere gevallen zijn de afmetingen van de uitgangsboon niet zeer groot. Volgens de classificatie van de bonenopbrengsten zijn er meestal zeer veel bonen in cl 8, en ook in cl 4, ook een enkele maal in cl 7. Er is enige overeenstemming tussen de samenstelling van de bonenopbrengst en de formule van de uitgangsboon.

We vinden onder het grote aantal gevallen van cl 8 met de formule 1 b th er enkele, waar de uitgangsboon voor de bonenopbrengsten in hoge mate de formule 1 b th in homozygote vorm heeft. We kunnen daarbij de formules 1 b th I, 1 b th II en 1 b th onderscheiden. Het is moeilijk, om de erfelijkheid van deze bonen vast te stellen. De invloed van het milieu is hier groot.

Summary.

The results are given of the F_4-seedgeneration of 1936, from initial beans of F_3-1935, that go back to crosses of 1934.

The material is grouped into 8 classes according to the slightly modified tetrahybrid scheme (1947; Proceed. Roy. Ac. of Sc., Amsterdam, Vol. 50, N. 7).

In a single case of cl 1a, according to the classification, the formula of the initial bean is $L_1 L_2$ B Th in the homozygous form to a high degree. In many cases the beanyield shows agreement with that of cl 2 with the form. L B th as of the I-line. Among the cases of cl 1b too there is a single one with the form. L B Th of the initial bean in an almost homozygous form. In many cases there are beans in many classes according to the classification, namely in cl 2, 4 and 8.

The beanyields of cl 1, form. L B Th show in many cases a relation to those of cl 2, form. L B th as of the I-line. According to the classification there are, besides a great number of beans in cl 2, also some beans in cl 1, 4 and 8 (tab. 5a).

TABLE 1. Some examples of the dimensions, weights and indices of F_3-startingbe
mean dimensions, weights and indices of the F_4-beanyields of 19

F_1-pl. of 1934 F_2-seed gener.	F_2-pl. of 1935 F_3-s. gen.	Flow. col. of F_2-pl.	Initial beans of F_3-1935	L	B	Th	W	L B	L Th	B Th	F_3-pl. of 1936 F_4-seed gener.	Flow. col. of F_3-pl.	n of beans
Cl 1a. The formula of the startingbeans is $L_1 L_2$ B Th.													
82	84	w	8 p 2 b	167	94	74	80	56	44	79	331	w	25
81	73	—	2 p 1 b	183	112	73	101	61	40	65	309	—	38
Cl 1 b. The formula of the startingbeans is $L_1 l_2$ B Th.													
81	70	—	1 p 2 b	136	98	68	65	72	50	69	298	—	24
66	47	v	17 p 5 b	135	90	69	61	67	51	77	194	w	50
81	78*)	—	17 p 1 b	152	95	60	62	63	39	63	321	w	22
65	43*)	v	5 p 6 b	126	85	68	43	68	54	80	165	—	38
Cl 2 b. The formula of the startingbeans is $L_1 l_2$ B th.													
82	87	—	6 p 2 b	141	106	59	60	75	42	56	337	w	24
82	95*)	w	11 p 2 b	152	98	72	74	65	47	74	361	w	25
83	101*)	w	7 p 4 b	121	81	63	44	67	52	78	1047	—	28
Cl 3. The formula of the startingbeans is L b Th.													
66	49	—	17 p 3 b	136	85	70	58	63	52	82	212	v	25
70	59*)	v	7 p 3 b	134	85	61	50	63	46	72	264	w	26
Cl 4. The formula of the startingbeans is L b th.													
83	101	w	4 p 3 b	131	84	64	50	64	49	76	1033	w	31
82	91*)	w	5 p 3 b	128	81	60	45	63	47	74	347	w	24
Cl 5. The formula of the startingbeans is l B Th.													
81	374	—	10 p 3 b	125	90	71	53	72	57	79	314	v	24
Cl 6. The formula of the startingbeans is l B th.													
66	49	—	4 p 1 b	128	94	63	53	73	49	67	203	w	28
Cl 7. The formula of the startingbeans is l b Th.													
66	51	—	2 p 3 b	119	79	68	47	66	57	86	984	w	22
83	105*)	—	16 p 1 b	138	89	70	62	65	51	71	400	w	27
Cl 8. The formula of the startingbeans is l b th.													
70	62	w	7 p 4 b	130	80	60	45	62	46	75	270	—	27
82	89	v	5 p 2 b	112	82	51	32	73	46	62	341	v	27
66	51	—	1 p 2 b	129	84	65	50	65	50	77	978	w	21
66	51	—	1 p 5 b	119	78	65	43	66	55	83	981	w	25
66	51	—	7 p 3 b	127	85	65	50	67	51	77	1000	v	25
83	101	w	7 p 5 b	109	75	59	34	69	54	79	1048	w	22
83	101	w	7 p 6 b	86	65	48	17	76	56	74	1049	—	10

*) The initial bean has in these cases another formula than that of the group in which it is

w = white, v = violet, p = pod, b = bean.

and the

TABLE 1a. Classification according to the simplified tetrahybrid scheme of the beanyields of 1936 of tab. 1.

W	L B	L Th	B Th	1 a $L_1 L_2$ B Th	1 b $L_1 l_2$ B Th	2 a $L_1 L_2$ B th	2 b $L_1 l_2$ B th	3 L b Th	4 L b th	5 l B Th	6 l B th	7 l b Th	8 a l b th I	8 b l b th II
74	59	43	72	12	5	1	4	0	3					
66	66	42	64	9	8	6	14	0	1					
71	65	47	71	1	20	0	3							
62	67	46	69	4	28	0	12	0	1	0	2	0	3	
66	64	46	72	1	15	0	4	0	0	0	1	0	1	
58	63	46	73		13	0	19	1	4	0	0	0	1	
60	64	43	68		4	1	18	0	0	0	1			
61	62	40	65	6	0	2	16	0	1					
54	62	44	69	1	0	0	15	0	5	0	0	0	7	
55	64	50	79	0	13	0	1	4	1	1	0	1	3	1
65	60	48	80		15	0	0	10	1					
41	63	39	61				10	0	10	0	0	0	11	
59	59	44	76	2	10	0	2	1	8	0	0	0	1	
51	68	53	77		2	0	0	1	1	2	3	6	8	1
54	68	53	78	0	8	0	1	2	0	9	1	4	1	2
42	71	58	81									6	9	7
49	67	54	80		5	0	0	1	0	8	0	7	4	2
31	63	49	77						2				25	
41	66	49	75				1				2		24	
39	63	50	79										18	3
38	65	53	82										6	19
40	65	49	75										24	1
37	62	44	72						2				20	
47	62	42	68				3		5		1		1	

Cl 2a, with the form. L_1 L_2 B th contains one case of which the initial bean of the beanyield is to a high degree homozygous for the form. L B th, as of the I-line, but the length contains a smaller number of L-factors as L L and perhaps also a smaller number of th-factors as th th.

Cl 2b with the form. L_1 l_2 B th contains some cases that agree with beanyields of the I-line, so where the initial bean is to a high degree homozygous for the form. L B th. Two cases correspond to the above mentioned case of cl 2a. There are also indistinct cases. The initial bean f.i. has, sometimes a great thickness, whereas according to the classification of the beanyield, the genotype answers the form. L B th. There are also beanyields with the form. L B th of the mean dimensions where the formula of the initial bean is l b th. We assume here that the genotype of the initial bean is L B th or L B Th. The phaenotype of the initial bean is l b th.

The beans of cl 3 have a small breadth. Mostly all three dimensions are not very great, owing to positive correlation and polymere heredity-factors. We find some heredity of small breadth.

The beans of cl 4 have the form. L b th. The length is mostly not very great. The beans are related to those of cl 2, form. L B th, the formula of the I-line. There are some good examples of this class in the material.

The beans of cl 5, form. l B Th are characterized by a small length and a high L B-index. They are related to the beans of the II-line by their great thickness. According to the classification, beans of the II-line have also beans in cl 5 and 8b (tab. 5b), besides many beans in cl 7. There is a single case with heredity of this beanform.

The beans of cl 6, form. l B th are little characteristic in our material; they closely agree with beans of cl 5. We describe such a case.

In the group of cases of cl 7, form. l b Th, as of the II-line we find some cases that wholly or almost wholly agree with beanyields of the II-line. The case too (pl. 984) that most closely approaches the II-line and that was extensively investigated is not yet identical with cases of the II-line.

The beans of cl 8 with the form. l b th may be subdivided in cl 8a, form. l b th I; cl 8b, form. l b th II, and cl 8c, form. l b th, in the strict sense. The group of cl 8 contains a great number of cases. The beans with the form. l b th can be compared with those with the form. L B Th. The dominance of the great dimensions over the small ones is only slight. There is much heterozygousness among these beans. Moreover the milieu-influence is great. It is difficult to establish the heredity here. We find some cases where the initial bean for the beanyield is more or less homozygous for the formula l b th. It is possible to distinguish here the formulas l b th I, l b th II and l b th.

So far we found no cases in our investigation, that wholly agree with beanyields of the I-line and of the II-line, i.e., of the two parent forms.

TABLE 2*a* The averages of comparison-beanyields of the I-line of 1935.

Pl	n	L	B	Th	W	LB	LTh	BTh
2	15	166	100	62	71	60	37	63
15	36	165	90	61	63	54	37	68
4	24	163	98	61	66	61	38	62
16	45	160	95	62	61	59	39	65
1	37	158	96	62	61	61	39	65
10	49	149	91	57	52	61	39	64
7	60	143	88	57	48	62	40	64
6	68	142	87	54	45	62	39	64
8	65	140	86	55	45	61	39	64

TABLE 2*b*. Of the II-line of 1935.

Pl	n	L	B	Th	W	LB	LTh	BTh
22	48	121	90	74	54	75	61	82
26	50	115	84	70	44	74	61	83
27	52	114	83	70	43	74	62	84
23	49	113	84	70	43	75	63	84
28	66	111	84	72	43	76	65	85
29	56	109	82	68	40	75	63	84
24	50	106	79	67	37	74	64	85
30	30	105	76	64	33	73	61	85

TABLE 3*a*. Comparison-beans of the I-line of 1935 with which initial beans of the F_3-generation of 1935 for the F_4-generation of 1936 agree.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
9	1p 1b	191	113	71	96	59	37	63
2	7p 1b	183	115	67	95	63	37	58
4	5p 2b	184	98	70	86	56	38	71
16	9p 2b	182	110	72	89	60	40	65
1	1p 3b	178	107	71	86	60	40	66
7	13p 2b	168	104	70	86	62	42	67
8	3p 2b	162	104	72	80	64	44	69
4	1p 3b	152	107	60	65	70	40	56
1	4p 3b	148	111	65	78	75	44	69
20	3p 2b	148	105	61	64	71	41	58
4	1p 4b	147	104	52	55	71	35	50

TABLE 3*b*. Comparison-beans of the I-line of 1935 for individual beans of F_3-beanyields of 1935, that agree with beanyields of the I-line of 1935.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
16	3p 2b	168	98	73	77	58	44	75
6	8p 1b	165	105	69	80	64	42	60
1	1p 2b	161	103	68	82	64	42	66
16	2p 3b	160	110	72	84	69	45	66
16	2p 2b	158	107	74	84	68	47	69
3	9p 1b	158	93	67	61	59	42	72
13	2p 2b	153	93	67	64	61	44	72
14	12p 2b	152	91	67	65	60	44	74
5	4p 3b	151	98	67	66	65	44	68
7	11p 2b	151	92	67	62	62	44	73

TABLE 4*a*. The averages of comparison-beanyields of the I-line of 1936.

Pl	n	L	B	Th	W	LB	LTh	BTh
81	24	155	95	61	63	61	40	64
28	25	151	91	62	62	60	42	69
89	24	149	93	63	61	62	42	68
42	27	149	90	59	57	61	39	66
70	25	145	86	62	56	59	43	73
40	25	143	89	57	52	63	40	65
39	25	137	85	61	51	62	45	72
68	25	130	77	57	41	60	44	74

TABLE 4*b*. Of the II-line of 1936.

Pl	n	L	B	Th	W	LB	LTh	BTh
113	36	120	87	73	51	73	61	84
114	21	118	85	69	47	72	58	81
108	20	116	87	70	48	75	60	80
115	22	114	86	72	47	75	63	84
101	46	112	84	71	46	75	63	85
116	20	112	81	66	40	72	59	81
102	20	109	78	63	37	72	58	81
103	22	108	81	68	42	75	63	84
96	20	106	79	65	40	74	61	82
119	20	103	77	63	33	75	61	82
97	23	101	75	61	33	74	60	81

TABLE 5*a*. Classification of beanyields of the I-line of 1936. Some examples.

Plant	Numb. of beans	1 a L_1L_2BTh	1 b L_1l_2BTh	2 a L_1L_2BTh	2 b L_1l_2BTh	3 LBTh	4 Lbth	5 lBTh	6 lBth	7 lbTh	8 a lbth I	8 b lbth II
81	24	4	0	10	9	0	1					
28	25	2	0	6	14	0	2				1	
42	27	2	0	2	19	0	4					
39	25		1	0	11	0	10				3	
80	27				8	0	9				10	
78	25				9	0	6		1		9	

TABLE 5*b*. Idem of the II-line of 1936.

Plant	Numb. of beans	1 a L_1L_2BTh	1 b L_1l_2BTh	2 a L_1L_2BTh	2 b L_1l_2BTh	3 LBTh	4 Lbth	5 lBTh	6 lBth	7 lbTh	8 a lbth I	8 b lbth II
113	36							26		10		
117	65							38		25	1	1
104	21							6		15		
101	45							13		27		5
106	21							1		18		2
105	20		1	0	0	0	0	0	0	13		6
97	23								1	4	2	16

TABLE 6. Some comparison-beans of the I-line of 1936 for individual beans of beanyields of F_4-1936, that agree with beanyields of the I-line of 1936.

Pl	Bean	L	B	Th	W	LB	LTh	BTh
93	2p 2b	180	104	73	80	58	41	70
93	2p 1b	167	102	70	75	60	42	69
43	3p 1b	166	99	74	86	60	45	75
43	3p 2b	163	97	71	82	60	44	73
59	4p 1b	160	95	74	71	59	46	78
43	1p 1b	159	89	69	66	56	43	78
42	2p 4b	153	101	69	70	66	45	68
29	7p 1b	151	87	67	65	58	44	77
89	4p 3b	150	96	69	73	64	46	72
89	4p 2b	145	94	70	72	65	48	75
39	5p 4b	142	88	68	62	62	48	77
37	5p 2b	141	95	67	67	67	48	71
92	3p 1b	136	89	65	57	65	48	73

TABLE 7*a*. Comparison-beans of the I-line of 1935 with a small length for initial beans of F_3-1935 for F_4-1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
5	202	125	82	43	27	66	34	53
6	242	124	84	57	39	68	46	68
	276	125	78	57	35	62	46	73
8′	15129	126	80	50	34	64	40	63
5	209	127	84	45	30	66	35	54
8′	15161	128	80	58	40	63	45	73
9	429	129	87	56	42	68	43	65
11	555	132	81	62	42	61	47	77
9	458	135	94	66	55	70	49	70

TABLE 7*b*. Comparison-beans of the II-line of 1935 for initial-beans of F_3-1935 for F_4-beanyields of 1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
22	4p 2b	130	92	72	59	71	55	78
22	3p 1b	128	93	77	61	73	60	82
22	2p 1b	127	95	77	61	75	61	85
25	19p 2b	127	86	67	48	68	53	78
23	11p 3b	126	85	66	45	68	52	78
23	11p 4b	122	82	64	41	67	53	78
25	13p 3b	120	82	63	41	68	53	77
24	11p 3b	119	83	63	40	70	53	76
25	19p 3b	118	82	69	43	70	58	84
25	10p 4b	115	76	62	36	66	54	82

TABLE 8. Exceptional beans of the II-line of 1936 to be compared to F_4-beans of 1936.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
113	1p 2b	130	91	73	62	70	56	80
113	1p 1b	129	91	72	57	71	56	79
114	4p 1b	125	89	69	51	71	55	78
108	3p 1b	123	90	72	55	73	59	80
116	3p 3b	121	87	66	45	72	55	76
108	1p 4b	120	87	68	48	73	58	78
108	1p 1b	117	85	67	46	73	57	79
114	5p 3b	115	83	66	44	72	57	80
105	1p 1b	114	85	67	45	75	59	79
117	11p 5b	107	80	62	37	75	58	78
97	2p 2b	121	87	59	44	72	49	68
119	1p 6b	97	74	58	28	76	60	58

TABLE 9. Examples of beans of F_3-1935 of various classes.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
cl 3, form. L b Th								
91	1p 4b	149	84	68	60	56	46	81
91	1p 2b	143	83	67	56	58	47	81
91	1p 3b	142	82	69	56	58	49	84
91	1p 1b	140	76	65	50	54	46	86
49	17p 3b	136	85	70	58	63	52	82
74	2p 1b	135	81	68	53	60	50	84
cl 4, form. L b th								
99	1p 1b	143	80	57	41	56	40	71
51	7p 4b	133	82	63	51	62	47	77
101	4p 3b	131	84	64	50	64	49	76
cl 5, form. 1 B Th								
33	16p 1b	130	92	68	55	71	52	74
74	12p 3b	125	90	71	53	72	57	79
33	6p 6b	122	88	69	49	72	57	78
33	6p 2b	121	92	69	50	76	57	75
33	6p 1b	118	91	71	49	77	60	78
cl 7, form. 1 b Th								
51	2p 1b	128	84	68	54	66	53	81
51	2p 2b	126	80	69	52	64	55	86
51	2p 4b	116	80	68	45	69	59	85
49	6p 1b	127	80	69	52	63	54	86
98	8p 3b	126	85	71	55	68	56	84

TABLE 10. F_4-1936. Beans of cl 5 form. 1 B Th, and of cl 8, form. 1 b th with the indices of cl 5.

Pl	Bean	L	B	Th	W	LB	L Th	B Th
314	3p 2b	125	87	67	55	70	54	77
	3p 1b	124	87	70	54	70	57	81
	3p 3b	121	86	66	51	71	55	75
	3p 5b [1])	122	87	65	52	71	53	75
	4p 5b	123	87	65	53	71	53	75
	3p 4b [2])	118	85	66	50	72	56	78
	2p 5b	117	84	65	48	72	56	77
	2p 3b	116	82	65	48	71	56	79
	2p 4b	115	85	63	47	74	55	74
	2p 2b	113	81	62	44	72	55	77
124	1p 1b	119	88	67	50	74	56	76
	1p 3b	118	87	66	49	74	56	76
	1p 2b [1])	114	86	63	43	75	55	73
	1p 4b	113	83	66	44	73	58	80
	1p 5b [3])	103	75	62	34	73	60	83

[1]) Form. cl 6. [2]) Form. cl 7. [3]) The last bean of the row in the pod.

Zoology. — *On the influence of various chlorides on maturation and cleavage of the egg of Limnaea stagnalis L.* By MIEK S. GRASVELD. (Zoological Laboratory, University of Utrecht). (Communicated by Prof. CHR. P. RAVEN).

(Communicated at the meeting of February 26, 1949.)

1. *Introduction.*

RAVEN and KLOMP (1946) described an abnormal cleavage of the eggs of *Limnaea stagnalis* in Ca-free media: the blastomeres remain spherical in shape and lose connection at an early stage; no cleavage cavity is formed; when cleavage advances, a loose aggregate of spherical cells is formed, surrounded by the vitelline membrane as by a loose envelope. Addition of $CaCl_2$ (minimum concentration 0.005 %) leads to normal cleavage; in 0.01—0.08 % $CaCl_2$ this effect is clearly observable. The primary action of the lack of Ca-ions seems to be a change in the properties of the vitelline membrane, which loses its contact with the egg surface.

OLGA HUDIG (1946) observed that the vitelline membrane of *Limnaea stagnalis* surrounds the egg from the very beginning till it is left by the embryo; initially it is very thin, at a later moment it becomes more distinctly visible. The membrane changes its consistency between the formation of the first and second polar body. It undergoes an alteration immediately after the egg is put into distilled water, probably through swelling; in consequence of this, the first polar body is situated inside the membrane and the "protoplasmic strand" which in normal eggs is seen between the first polar body and the egg and, later, between the first and second polar body is not visible.

This can be explained by supposing that the first polar body is always inside the membrane, which is pulled tightly around it in normal media. This view is supported by the observation that the first polar body always comes to lie inside the membrane, when eggs are transferred to distilled water. On the other hand, the first polar body of normal eggs detaches itself easily from the egg surface, when the eggs are stirred; this speaks strongly against the enclosure of the first polar body by the membrane; the connecting strand might be a spindle remnant.

The vitelline membrane does not swell in a mixture of distilled water and $CaCl_2$. In solutions of 0.2—0.0125 % $CaCl_2$ the first polar body is mostly situated outside the membrane, in 0.00625 % and less concentrated solutions inside the membrane. In 0.04—0.005 % $CaCl_2$ cleavage is normal; in the latter concentration the blastomeres do not join quite so closely as they do in the higher concentrations and the cleavage cavity is formed late.

The vitelline membrane must be considered to be a chorion; it does not take an active part in the cleavage and does not form the cell walls. It is concluded that not only the membrane, but also a protoplasmic cortex layer, lying immediately beneath it, is influenced by Ca-ions.

In KCl cleavage may be normal, perhaps because the K-ions liberate the Ca-ions, present in the cortex.

RAVEN and MIGHORST (1946) studied the influence of highly concentrated $CaCl_2$-solutions on maturation in the egg of *Limnaea stagnalis*. No polar bodies are formed in 3 % and 1,5 % $CaCl_2$. One small, spindle-shaped polar body is sometimes formed in 1.1 % $CaCl_2$; a larger percentage of the eggs forms a first polar body and sometimes a second polar body in 0.75 %. In 0.5 % $CaCl_2$ depolarization of the second maturation spindle occurs in some egg-masses. In 0.4 % nearly all eggs develop to the 2-cell stage. In 0.2 % a morula stage may be attained.

In higher concentrations small clear spots appear in the vitelline membrane; after some hours small vesicles have been formed on the surface of the egg in these places; still later the membrane is lifted locally from the egg surface, forming large blebs, in which the polar bodies are sometimes lying. This forms an argument in favour of the view that the polar bodies lie inside the membrane. The eggs can accumulate Ca-ions from a concentrated $CaCl_2$-solution.

DE GROOT (1948) treated the eggs of *Limnaea stagnalis* with LiCl-solutions, varying from 0.05—4 %. Eggs, transferred to a 4 % solution immediately stop their development and cytolysis occurs. In 2 % a partly extruded polar body is formed; in 1.5 and 1 % its formation is prevented, when treatment begins more than 30—35 minutes before; if it begins at a later moment the first polar body is deformed; in 0.6 % and 0.5 % a first polar body is always extruded, mostly deformed; in 0.4 % a second polar body may be formed during intensive amoeboid movements of the egg; in 0.35—0.05 % both polar bodies are nearly always formed; 0.35 % is the highest concentration, in which a cleavage furrow is observed, appearing with considerable delay; in 0.05 % there is no delay and sometimes a second cleavage occurs.

The observed effects in concentrations above 0.2 % need not to be due to a specific influence of LiCl, but may be caused primarily by hypertonicity of the medium, if the extrusion of polar bodies has to be understood as a process of osmotic activity.

In 0.1 % and 0.05 % the blastomeres flatten themselves against each other and a cleavage cavity is formed. Hence, not only $CaCl_2$ but also LiCl is able to cause flattening.

It is likely that the vitelline membrane is affected by LiCl.

In the following investigation the influence of $CaCl_2$, $MgCl_2$, LiCl, NaCl, and KCl on maturation and cleavage of the egg of *Limnaea stagnalis* has been compared. The position of the polar bodies, the behaviour of the vitelline membrane and the type of cleavage were especially observed.

2. *Material and methods.*

Eggs were obtained in the usual way by stimulation of the snails with *Hydrocharis* (cf. RAVEN and BRETSCHNEIDER 1942).

Each egg-mass was divided into 6 parts; the eggs were decapsulated immediately after oviposition and washed in distilled water in order to remove the capsule fluid; 10 eggs of each egg-mass were transferred to equimolar solutions of $CaCl_2$, $MgCl_2$, LiCl, NaCl and KCl, respectively.

Stock solutions of 0.1 M. were made from the crystallized chlorides; they were diluted with distilled water. With $CaCl_2$ and $MgCl_2$, which contain crystal water, the percentages given are computed to net weights.

The eggs were put in a small drop of the solution on a hollow slide, on which the cover glass was fixed on two sides with paraffine; in this way, the preparations can be placed vertically on a horizontal microscope in order to study the polar bodies at the animal pole, which is, as a rule, turned upwards. To get a clear view of the vitelline membrane, it is necessary to close the diaphragm of the microscope a great deal. In the long run, the eggs beneath the cover glass do not get sufficient oxygen and, after a day, abnormalities ensue and development stops. The control eggs developed inside the egg capsules. All eggs were kept at a temperature of 25° C in a water bath.

Table 1 gives a general survey of the experiments.

TABLE I. General survey of the experiments.

Solutions	$CaCl_2$			$MgCl_2$			LiCl			NaCl			KCl		
Concentration (Mol.)	Conc. (%)	Relative osmotic pressure [1]	Number of eggmasses [2]	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses	Conc. (%)	Relative osmotic pressure	Number of eggmasses
0,1	1,1	3	10	0,95	3	8	0,42	2	11	0,58	2	9	0,75	2	8
0,05	0,55	$1^1/_2$	12	0,475	$1^1/_2$	14	0,21	1	15	0,29	1	17	0,375	1	17
0,025	0,275	$^3/_4$	9	0,2375	$^3/_4$	9	0,105	$^1/_2$	9	0,145	$^1/_2$	9	0,1875	$^1/_2$	9
0,0125	0,1375	$^3/_8$	5	0,1188	$^3/_8$	5	0,0525	$^1/_4$	5	0,0725	$^1/_4$	5	0,0938	$^1/_4$	5
0,00625	0,0688	$^3/_{16}$	9	0,0594	$^3/_{16}$	9	0,0262	$^1/_8$	6	0,0363	$^1/_8$	6	0,0469	$^1/_8$	6
0,00312	0,0344	$^3/_{32}$	8												
0,00156	0,0172	$^3/_{64}$	6	0,0148	$^3/_{64}$	1	0,0066	$^1/_{32}$	1	0,0091	$^1/_{32}$	1	0,0117	$^1/_{32}$	1
0,00078	0,0086	$^3/_{128}$	2												
0,00039	0,0043	$^3/_{256}$	4												
0,00019	0,0022	$^3/_{512}$	1												

[1]) Relative osmotic pressure $= \frac{\text{osm. pr. of solution}}{\text{internal osm. pr. of eggs}}$

[2]) On an average, 10 eggs pro egg-mass were treated with each solution.

3. *Results of experiments.*

A. *Normal development and influence of distilled water.*

The development of the eggs in the capsules has been described by O. Hudig (1946). Some additional observations have been made:

1. The "protoplasmic strand" between first and second polar body does not always disappear, but it becomes smaller in all cases.

2. At the 24-cell stage and later the second polar body is no longer flattened, but spherical in shape and swollen; the first polar body is no longer spherical, but shows an irregular outline.

The development in distilled water has also been observed by O. HUDIG; she describes that the membrane takes part in the movements of the egg surface during cleavage. This cannot be the case at the animal pole, however, for both polar bodies are distinctly visible here inside the membrane. Moreover, the bridge at the vegetative pole is formed already before the cleavage groove is complete. The first polar body has a less distinct outline than the second polar body in distilled water.

B. *The development in the various chlorides.*

a. The stage of development, reached in the different media, is shown in table 2. The results are arranged according to the relative osmotic

TABLE 2. The stage of development reached in the different media.

Rel. osm. press.	$CaCl_2$	$MgCl_2$	LiCl	NaCl	KCl
3	30 % 1st p.b.	15 % 1st p.b.			
2			50 % 1st p.b. 15 % 2nd p.b.	50 % 1st p.b. sometimes 2nd p.b.	35 % 1st p.b. sometimes 2nd p.b.
$1^1/_2$	100 % 1st p.b. 50 % 2nd p.b.	100 % 1st p.b. 50 % 2nd p.b.			
1			4-8-c. st. (delay)	morula-st.	12-24-c. st.
$^3/_4$	16-24-c. st.	24-c. st.			
$^1/_2$			4-c. st.	16-24-c. st.	16-24-c. st.
$^3/_8$	16-c. st.	morula-st.			
$^1/_4$			2-4-c. st.	8-c. st.	4-8-c. st.
$^3/_{16}$	16-c. st.	8-16-c. st.			
$^1/_8$			2-4-c. st.	4-8-c. st.	4-c. st.
$^3/_{32}$	16-c. st.				
$^3/_{64}$	8-c. st.	4-16-c. st.			
$^1/_{32}$			4-c. st.	4-c. st.	4-c. st.
$^3/_{128}$	12-c. st.				
$^3/_{256}$	8-c. st.				
$^3/_{512}$	4-c. st.				

Abbrev.: p.b. = polar body; st. = stage.

pressures of the salt solutions as compared with the internal osmotic pressure of freshly-laid eggs, which equals that of a 0.09—0.10 M. solution of a non-electrolyte according to RAVEN and KLOMP (1946). Hence, the eggs of *Limnaea stagnalis* are about isotonic with 0.03 M. $CaCl_2$ and $MgCl_2$, 0.045—0.05 M. LiCl, NaCl and KCl.

In judging the results, it has to be kept in mind that in all solutions after some time development stops in consequence of the lack of oxygen. As a rule, this occurs at the 16—24 cell stage. Hence, the data of table 2 have only a relative value.

In solutions of $CaCl_2$ and $MgCl_2$ which are highly hypertonic with regard to the eggs, the first polar body is rarely, the second polar body

never formed. With decreasing hypertonicity, the number of first polar bodies rises and also second polar bodies appear. Finally, in solutions which are only slightly hypertonic, the first polar body is formed in 100 %, the second polar body in 50 % of cases in $CaCl_2$ and $MgCl_2$.

In iso- and hypotonic solutions of all chlorides both polar bodies are always formed. Development stops after extrusion of the polar bodies in hypertonic solutions; in isotonic and hypotonic solutions cleavage occurs in all chlorides.

The highest stage of development is reached in 0.025—0.00312 M. $CaCl_2$ and $MgCl_2$, and 0.05—0.025 M. LiCl, NaCl and KCl. In less concentrated solutions development stops sooner. In 0.00019 M. $CaCl_2$ and 0.00156 M. KCl and NaCl development stops at the same moment as in distilled water.

One may conclude that hypertonicity of the solutions causes interruption of maturation, while for further development an optimum exists at a moderate degree of hypotonicity. This agrees with the results of RAVEN and KLOMP (1946), HUDIG (1946), RAVEN and MIGHORST (1946) and DE GROOT (1948).

In none of the LiCl solutions development proceeded beyond the 4—8 cell stage. Hence, LiCl seems to be the most noxious of these chlorides.

b. Further effects of hypertonic solutions.

1. *Rotation of the eggs.*

When the eggs are in the capsule fluid, they have a tendency to turn their animal pole upwards. This does not occur in hypertonic solutions, viz. in 0.1 M. $CaCl_2$ and $MgCl_2$, 0.05 M. $CaCl_2$ and in 0.1 M. LiCl, NaCl and KCl.

2. *Shape of the first polar body.*

Deformed first polar bodies were observed in hypertonic solutions, viz. in 0.1 M. and 0.05 M. $CaCl_2$ and $MgCl_2$ and in 0.1 M. LiCl, NaCl and KCl. As a rule, they are round at first, then they become pear-shaped (after $\frac{1}{4}$—$\frac{1}{2}$ hour) and still later pointed, with a conical protuberance at the top (fig. 1). The deformation in $MgCl_2$ is greater than in $CaCl_2$, in

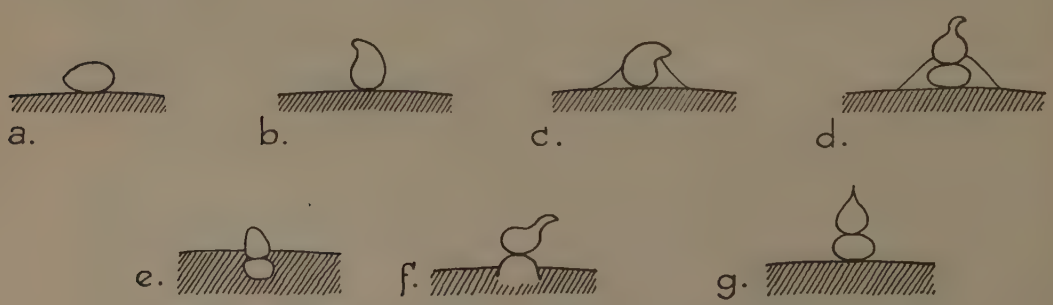

Fig. 1. Deformation of first polar body in *a*) 0.1 M. $CaCl_2$; *b*) 0.05 M. $CaCl_2$; *c*) 0.1 M. $MgCl_2$; *d*) 0.05 M. $MgCl_2$; *e*) 0.1 M. LiCl; *f*) 0.1 M. NaCl; *g*) 0.1 M. KCl.

NaCl greater than in KCl, in KCl greater than in LiCl. The polar bodies are normally shaped in all other solutions. Evidently, water is withdrawn from the polar bodies in hypertonic solutions.

Deformed polar bodies have been observed in hypertonic solutions of LiCl by DE GROOT (1948).

3. *Vesicles and blebs at the egg surface.*

Lipoid-looking vesicles and blebs at the egg surface appear in hypertonic solutions. Such vesicles were described as clear spots in the vitelline membrane, which forms great blebs at a later moment, by RAVEN and MIGHORST (1946).

Probably, the vesicles change into blebs; partly the vesicles remain, however; may-be they are vacuoles, containing a lipid substance; they are situated, however, inside and not in the membrane (fig. 2).

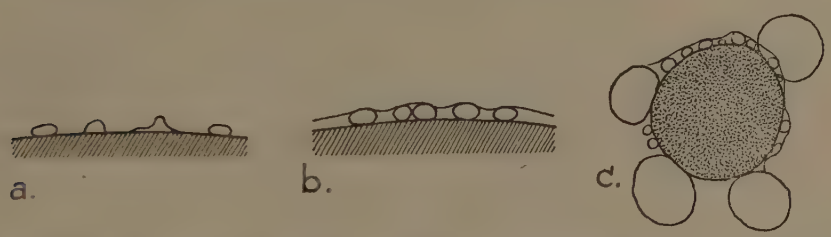

Fig. 2. Formation of "lipoid" vesicles and blebs on the egg-surface in 0.1 M. $CaCl_2$ after *a*) 6 hours, *b*) 9 hours, *c*) 2—3 days.

4. *Amoeboid movements in hypertonic LiCl-solutions.*

The eggs show considerable amoeboid movements in 0.1 M. LiCl 3—7 hours after the first cleavage in the control eggs. They show the same movements in 0.05 M. LiCl after the extrusion of the first polar body and sometimes after the first cleavage.

DE GROOT also describes a strong amoeboid activity in hypertonic LiCl-solutions. As no increase of amoeboid activity of the eggs has been observed in the other chlorides, this seems to be a specific Li-effect.

c. Position of polar bodies.

In the following description the term of O. HUDIG "polar bodies outside the membrane" is used; this means a position of the first polar bodies as in control eggs.

Attention has been paid to:

1. The "protoplasmic strand" (O. HUDIG).
2. The vitelline membrane.
3. The flattening of the polar bodies.

Table 3 gives a survey of the results. They can be summarized in this way:

TABLE 3. The position of the 1st polar bodies.

Concentr. (Mol.)	$CaCl_2$	$MgCl_2$	LiCl	NaCl	KCl
0,1	vitelline membrane not visible				
0,05	± normal	inside	inside	inside	inside
0,025	± normal	normal	inside	inside	inside
0.0125	inside	normal	inside	inside	inside
0,00625	normal	normal	inside	inside	inside
0,00312	normal				
0,00156	normal	inside	inside	inside	inside
0,00078	inside				
0,00039	inside				
0,00019	inside				

1. In LiCl, NaCl and KCl solutions of all concentrations studied both polar bodies are situated inside the membrane.

2. In concentrated solutions of $CaCl_2$ and $MgCl_2$, both polar bodies are inside the membrane or the latter is passing directly from the egg surface toward the first polar body.

3. In weaker solutions of these salts, the position of the polar bodies resembles that in normal eggs (first polar body outside, second polar body inside membrane); most so in 0.0625—0.00156 M. $CaCl_2$ and 0.025—0.00625 M. $CaCl_2$. In these cases, a distinct "protoplasmic strand" is present between first polar body and egg, later between both polar bodies.

4. In still weaker solutions, it is obvious that this "protoplasmic strand" actually is the vitelline membrane, which passes now to the outside of the first polar body, so that both polar bodies are inside the membrane.

d. The type of cleavage.

Attention has been paid to:

1. Flattening of the blastomeres.
2. Formation of membrane bridges between the blastomeres.
3. Formation of a cleavage cavity.

Table 4 gives a survey of the types of cleavage obtained in the various solutions. From the observations, the following conclusions can be drawn:

1. Though no absolute correlation between the 3 phenomena exists, in general there is a direct relation between the degree of flattening and the development of the cleavage cavity, whereas an inverse relation exists between both these phenomena and the formation of membrane bridges at cleavage. The only exception to this rule is formed by cleavage in 0.05 M. LiCl, where a high degree of flattening is not attended with the formation of a cleavage cavity.

2. The cleavage type is most normal in $CaCl_2$, $MgCl_2$ and LiCl at a moderate degree of hypotonicity (0.0125—0.00312 M.), somewhat less in NaCl at this same concentration, very abnormal in KCl. At both higher

Fig. 1. Sperm head with head cap after fixation in mercurochrome; orig. magn. 2500 ×.
Fig. 2. Sperm head with head cap after fixation in mercurochrome; orig. magn. 10000 ×.
Fig. 3. Sperm head with vacuoles; orig. magn. 12500 ×.

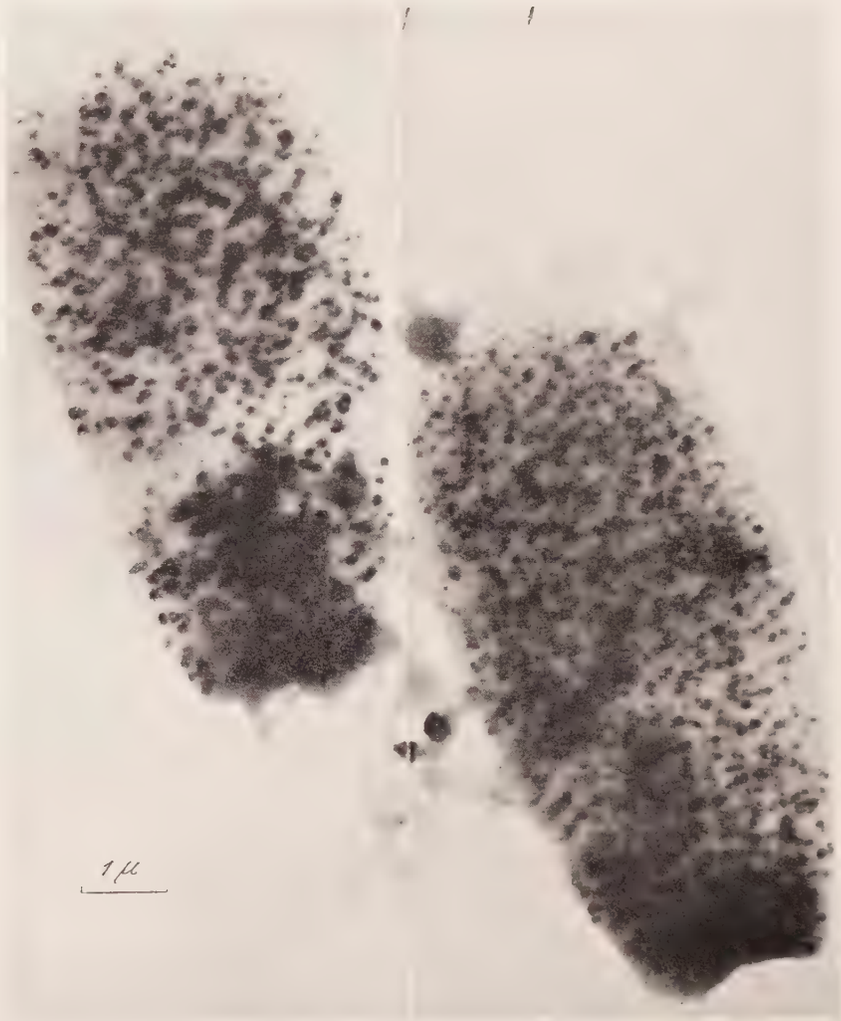

Fig. 4. Sperm head after hydrolysis and $AgNO_3$ impregnation; orig. magn. 30000 ×.

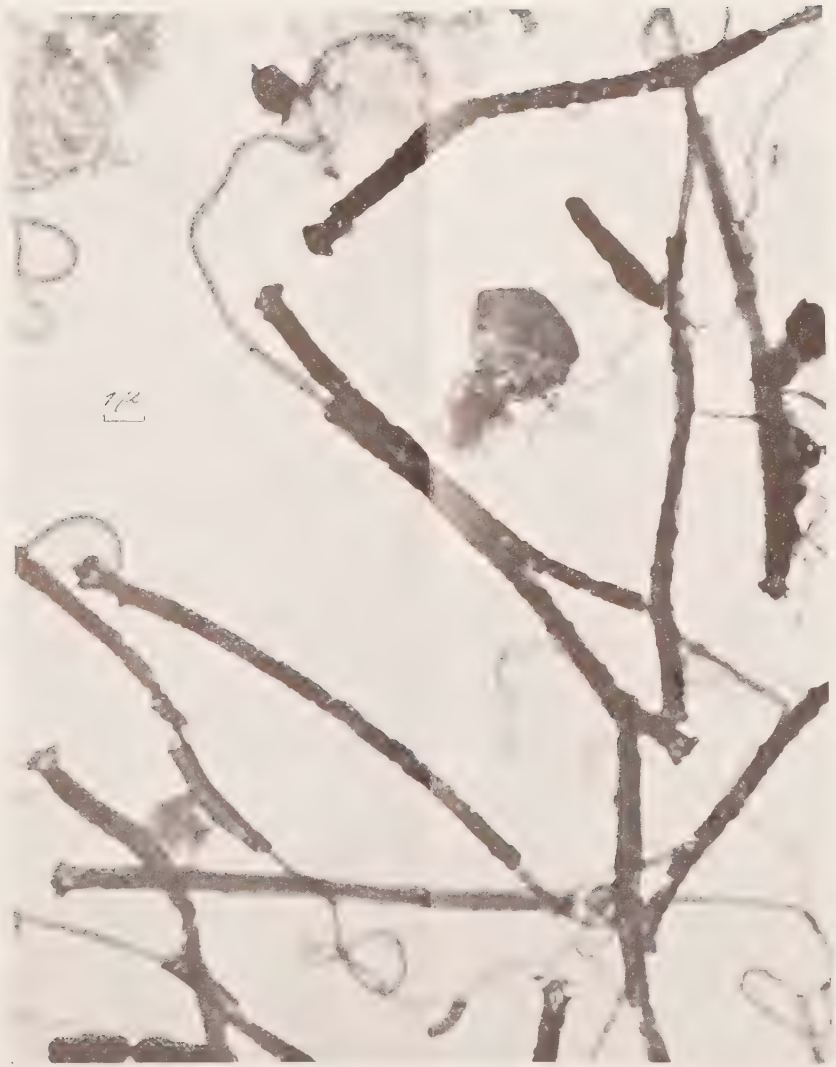

Fig. 5. Middle piece with spiral body, after bacterial maceration and impregnation with chromic acid. Orig. magn. 8000 ×.

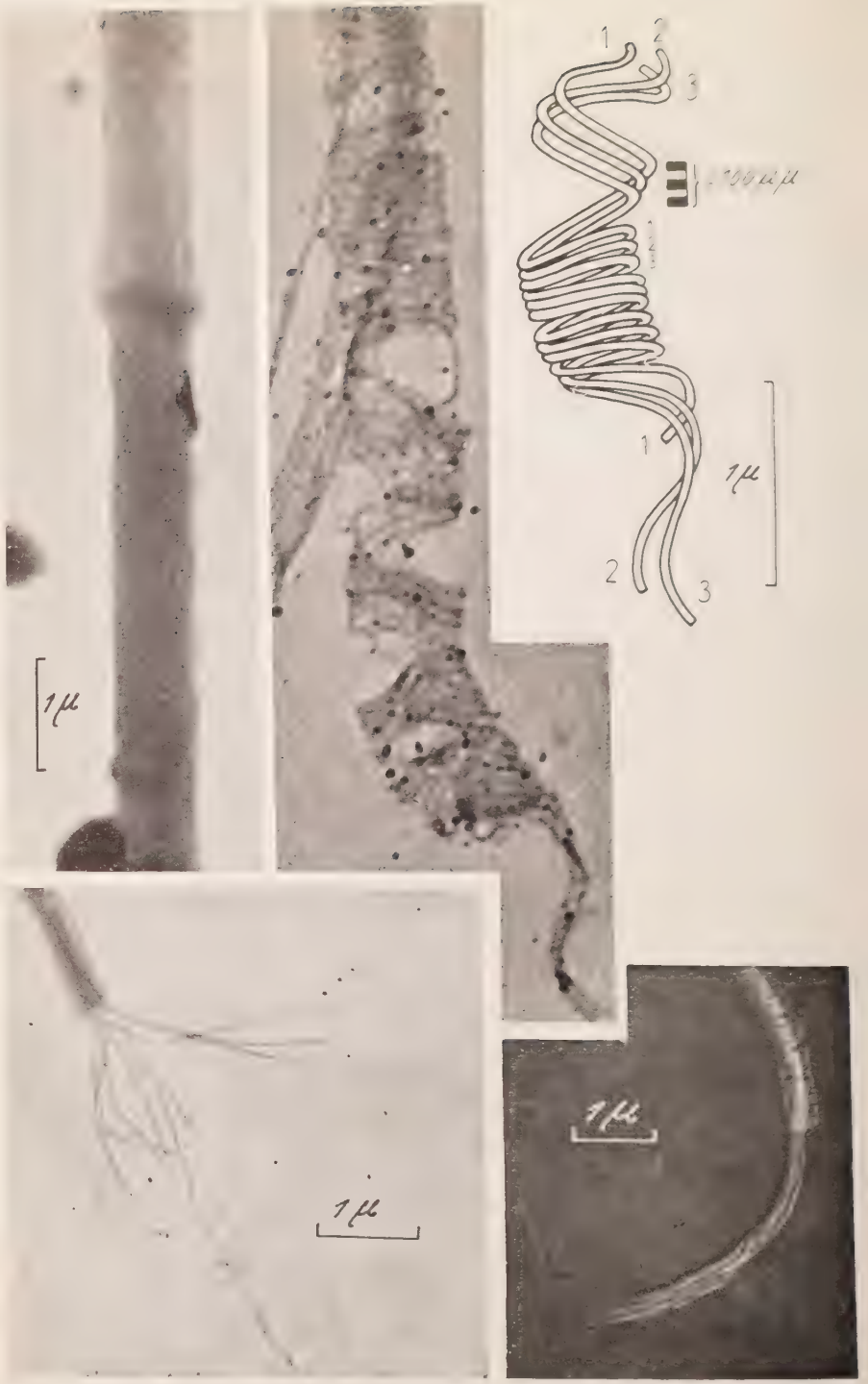

Fig. 6. JENSEN's ring after $AuCl_4$ impregnation. Orig. magn. 17000 ×.

Fig. 7. Subfibrils and cortical spiral of tail, after maceration and OsO_4 impregnation. Orig. magn. 36000 ×.

Fig. 8. Terminal piece of tail after chloramine; orig. magn. 15000 ×.

Fig. 9. Terminal piece of tail after shadow-casting; orig. magn. 12000 ×.

TABLE 4. Cleavage in the different chlorides.

Conc. (Mol.)	$CaCl_2$			$MgCl_2$			LiCl			NaCl			KCl		
	F	*C*	*B*	*F*	*C*	*B*	*F*	*C*	*B*	*F*	*C*	*B*	*F*	*C*	*B*
0,05							+++*D*	−	+	+	−	+	−	−	+++
0,025	+	−	+	+	−	−	+++*D*	++*D*	+	+	+	++	+	−	+++
0,0125	++	+++	−	+++	+++	−	+++*D*	+++*D*	+	++	+++	+	+	−	+++
0,00625	+++	+++	−	+++	+++	−	+++*D*	+++*D*	+	++	+++	+	+	−	+++
0.00312	+++	+++	−												
0,00156	++	++	−	+	+*D*	+	−	−	+++	−	−	+++	−	−	+++
0,00078	+	++*D*	−												
0,00039	+	+	+												
0,00019	±	±	++												

Abbrev.: *F* = Flattening of the blastomeres.
C = Formation of cleavage cavity.
B = Formation of membrane bridges.
The number of plus marks denotes the strength of the phenomenon (+ = slight, ++ = moderate, +++ = strong).
D = delayed.

and lower concentrations the flattening of the blastomeres and the formation of a cleavage cavity are suppressed.

3. The degree of flattening decreases in the order LiCl—NaCl—KCl.

4. The incidence of membrane bridges increases in the order $CaCl_2$ and $MgCl_2$—LiCl—NaCl—KCl.

5. The flattening of the blastomeres can be preceded by formation of a bridge at the vegetative pole, but the latter disappears during the flattening. If flattening does not occur, the bridge is permanent.

6. The polar bodies are mostly inside the membrane, when bridges are formed and no flattening and cleavage cavity are observed then.

7. Both the flattening of the blastomeres and the formation of a cleavage cavity are delayed in LiCl. The second cleavage shows a delay of $\frac{1}{2}$—2 hours in LiCl solutions.

e. *Abnormal cleavage.*

In some cases the eggs cleave into three blastomeres: the first cleavage is a little delayed, three blastomeres with three nuclei are formed. Secondarily, two of these cells may fuse, so that a 2-cell stage is formed, one of the cells containing two nuclei. The second cleavage of 3-cell stages, which is somewhat delayed, leads to a six-cell stage, which does not flatten, and forms membrane bridges; then development stops.

Though this phenomenon only occurs in cultures which are cut off from the air, it attracts attention that it occurs more frequently in some solutions (table 5), chiefly in $CaCl_2$ and $MgCl_2$, seldom in KCl and NaCl, never in LiCl.

It is evident that the egg is most susceptible to lack of oxygen in $CaCl_2$ and $MgCl_2$.

TABLE 5. Abnormal cleavages in various chlorides.

Conc. (Mol.)	$CaCl_2$		$MgCl_2$		LiCl		NaCl		KCl	
	Number of eggs	Number of abn. cleav.	Number of eggs	Number of abn. cleav.	Number of eggs	Number of abn. cleav.	Number of eggs	Number of abn. cleav.	Number of eggs	Number of abn. cleav.
0,05	—	—	—	—	150	0	170	9	170	seldom
0,025	90	3	90	4	90	0	90	0	90	1
0,0125	50	8	50	4	50	0	50	0	50	2
0,00625	90	16	90	18	60	0	60	0	60	0
0,00312	80	2	—	—	—	—	—	—	—	—
0,00156	60	2	10	0	10	0	10	0	10	0
0,00078	20	1	—	—	—	—	—	—	—	—
0,00039	40	6	—	—	—	—	—	—	—	—
0,00019	10	1	—	—	—	—	—	—	—	—
Total	440	39	240	26	360	0	380	9	380	>3
%	9		11		0		2		± 1	

4. *Discussion.*

Considering the results of my experiments, it is clear that the salt solutions studied have influenced 3 groups of factors:

1. the consistency of the vitelline membrane, which determines the position of the polar bodies with respect to the latter;

2. the "adhesiveness" of the vitelline membrane to the egg surface, determining its behaviour during cleavage;

3. the properties of the egg cortex, responsible for the flattening of the blastomeres and, indirectly, for the formation of the cleavage cavity.

1. It is evident from my observations that the protoplasmic strand mentioned by O. HUDIG is not a spindle remnant, but the same as the membrane, which undergoes an alteration between the formation of the first and second polar body, as supposed by RAVEN (1945). Normally the membrane is almost liquid at the formation of the first polar body; it is situated closely around it and is visible as a "protoplasmic strand" between the first polar body and the egg. This strand shrinks at a latter moment, probably by hardening; in consequence of this the first polar body remains attached to the egg surface. I never observed (as O. HUDIG did) that it can be detached from the egg surface by stirring the cover glass. The membrane gets another consistency before the formation of the second polar body; in consequence of this it is distinctly visible as a pellicle, tightened over the second polar body, the latter being flattened by it.

In some concentrations of $CaCl_2$ and $MgCl_2$ this process occurs likewise. In distilled water, LiCl, KCl and NaCl the change in the consistency of the membrane takes place at an earlier stage than in the control eggs; the same occurs in the less concentrated solutions of $CaCl_2$ and $MgCl_2$; in consequence of this there is no "protoplasmic strand" and the membrane

is visible as a pellicle. The normal course of the process requires a minimum concentration of $CaCl_2$ or $MgCl_2$; it never occurs in LiCl, NaCl and KCl.

Considering the protoplasmic strand as membrane, one can conclude that the first polar body is always inside the membrane. This agrees with the fact that in some cases polar bodies, which appear to lie outside the membrane at first cleavage are found inside at the 24-cell stage.

The swelling of the second polar body in the control eggs also points to the fact that the membrane, tightened over it, has another consistency than near the first polar body; the latter, surrounded by the hardened membrane, is irregular in shape.

Although the membrane is not immediately visible, it encloses the egg from the very beginning. This may be concluded from:

a. Wrinkling of the membrane in distilled water and low salt concentrations (0.00039 M. $CaCl_2$, 0.00156 M. LiCl).

b. Lifting of the membrane by the lipoid-looking vesicles in hypertonic solutions.

c. The existence of the protoplasmic strand or the pellicle after the formation of the first polar body.

2. The vitelline membrane does not take an active part in cleavage, for cleavage continues when membrane bridges are formed; the walls of the blastomeres cannot be formed by the membrane, for it does not follow the cleavage furrows.

The behaviour of the membrane at cleavage parallels the position of the polar bodies. The accelerated change of the membrane consistency by lack of 2-valent kations, which causes a position of the polar bodies inside the membrane, is attended with formation of bridges between the blastomeres. However, there is no strict correspondence between both phenomena. Whereas the changes in consistency of the membrane, as evidenced by the position of the polar bodies, are alike in LiCl, NaCl and KCl, the incidence of membrane bridges in these solutions increases in the order mentioned. This points to a difference between the properties of the membrane which have been called "consistency" and "adhesiveness", respectively.

3. The changes in cleavage type, brought about by a diminished flattening of the blastomeres, which leads to a suppression of the cleavage cavity, may be explained by an influence of the salts on the properties of the egg cortex. Normal cleavage may occur in $CaCl_2$, $MgCl_2$ and LiCl of optimal (moderately hypotonic) concentrations; in NaCl cleavage type is less normal; in KCl it is very abnormal.

It might be supposed that among the properties of the egg cortex involved its viscosity and "tension at the surface" play a prominent part. In table 6, our results have, therefore, been compared with those of DE VRIES (1947) on the influence of $CaCl_2$ and LiCl on these properties of the egg, as determined with the centrifuge method. The table shows, however, that no correspondence between both groups of observations exists. As a matter of fact, the centrifuge method yields only "gross"

TABLE 6. Comparison with the results of DE VRIES (1947).

$CaCl_2$					LiCl				
Mol.	Viscosity	Tension	Flattening	Cleavage cavity	Mol.	Viscosity	Tension	Flattening	Cleavage cavity
0,12	—	+			0,25	—	+		
0,04	—	0			0,15	—	+		
0,025			+	—	0,1				
0,0125			++	+++	0,05			+++	—
0,009	—	0			0,025	—	—	+++	++
0,008	0	0			0,02	0	0		
0,00625			+++	+++	0,015	0	0		
0,005	0	0			0,0125	0	0	+++	+++
0,0045	+	0			0,0075	0	0		
0,00312			+++	+++	0,00625			+++	+++
0,003	0	0			0,005	0	0		
0,002	0	0			0,0025	+	0		
0,00156			++	++	0,002	0	0		
0,00078			+	++	0,00156			—	—
0,0007	0	0			0,0015	0	0		
0,00039			+	+					
0,00019			±	±					

Decrease of viscosity (tension): —
Increase „ „ „ : +
Flattening and cleavage cavity cf. table 4.

results, whereas in cleavage local changes in both properties may play a predominant part. This might explain the lack of correspondence between both experiments.

4. The question arises if either the various chlorides affect specifically maturation and cleavage of the egg of *Limnaea stagnalis*, or the influence only depends on the number of ions present.

Hypertonicity of the medium stops development at an early stage, prevents normal rotation of the eggs and suppresses formation of polar bodies; it causes formation of vesicles and blebs at the egg surface. If polar bodies are extruded, they are deformed.

These phenomena are not characteristic for the presence of special ions. In hypotonic solutions, however, characteristic differences between the ions become evident.

In $CaCl_2$ and $MgCl_2$ solutions of optimal (moderately hypotonic) concentrations, development is most normal. The consistency and adhesiveness of the vitelline membrane resemble those of normal eggs in egg capsule fluid, cleavage type is normal with good flattening of the blastomeres and well-developed cleavage cavity. The eggs are, however, rather susceptible to lack of oxygen.

In LiCl, hypertonic solutions cause a great amoeboid activity. In all

concentrations, development is scanty and does not proceed beyond the 8-cell stage. The consistency of the vitelline membrane is abnormal, but its adhesiveness only slightly affected. Cleavage type may be normal in optimal concentrations.

NaCl allows of a good development, but causes an abnormal consistency of the membrane. The adhesiveness of the latter is more affected than in LiCl and cleavage type is not entirely normal.

In KCl-solutions, a good development is possible, but both the consistency and the adhesiveness of the vitelline membrane are heavily affected and the cleavage type is very abnormal.

5. *Summary.*

1. Hypertonic solutions of $CaCl_2$, $MgCl_2$, LiCl, NaCl, and KCl interrupt maturation.

2. Deformation of the first polar body occurs in hypertonic solutions of all chlorides used; normal polar bodies are formed in iso- and hypotonic solutions.

3. Hypertonic solutions of all chlorides used cause formation of vesicles and blebs at the egg surface, inside the membrane.

4. The first polar body is always inside the membrane, which undergoes a change in normal development between the formation of the first and second polar body; it is almost liquid during the formation of the first polar body, it is tightened as a pellicle over the second polar body.

5. The change in consistency of the membrane occurs earlier by lack of bivalent kations.

6. The "adhesiveness" of the vitelline membrane to the egg surface is diminished by lack of bivalent kations; this is most evident in KCl, somewhat less in NaCl, still less in LiCl.

7. The cleavage type may be normal in $CaCl_2$, $MgCl_2$ and LiCl; less normal it is in NaCl; very abnormal in KCl.

8. Hypertonic LiCl solutions cause a great amoeboid activity of the eggs.

9. The eggs are most susceptible to lack of oxygen in $CaCl_2$.

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Zoology. — *The effect of a short treatment with thiourea upon the fish thyroid gland.* By J. LEVER, J. MILTENBURG and G. J. VAN OORDT. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht, and Station Biologique de Roscoff, France.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of February 26, 1949.)

The effect of thiourea upon the fish thyroid has been shown for the first time by GOLDSMITH, NIGRELLI, GORDON, CHARIPPER and GORDON (1944), who found that immersion of fish of a hybrid strain of *Xiphophorus helleri* and *Platypoecilus maculatus* in 0.033 — 0.67 % solutions of this antithyroid drug resulted in hyperplasia of the thyroid gland, which was first observed after 49 days of treatment; the number of follicles increased and cell proliferations took place. This was mentioned also by CHARIPPER and GORDON (1947).

Similar results were reported by NIGRELLI, GOLDSMITH and CHARIPPER (1946) after having treated guppies (*Lebistes reticulatus*) with a 0.03 % solution of thiourea for about 4 months.

In view of these results it became of interest to ascertain the effect of a short treatment with thiourea upon the thyroid glands not only of a freshwater, but also of a marine fish, viz. *Lebistes reticulatus* and *Callionymus lyra*, respectively.

Immediately after parturition adult females of *Lebistes reticulatus* were placed separately in glass-bowls, containing 200 cc of tapwater, in which 0.01, 0.1, or 0.6 % thiourea had been dissolved. The last mentioned concentration was highly toxic, however. All glass-bowls were kept in a room with a constant temperature of 24° C.

The specimens of *Callionymus lyra*, having a length of about 9 cm, were kept at approximately 16° C. separately in glass-bowls, containing 750 cc of seawater, in which thiourea was dissolved in concentrations of 0.01 or 0.1 %, the latter being also highly toxic. These solutions were changed every 2 days.

Untreated controls were kept in both cases under similar conditions.

The *Lebistes*-heads and the thyroid glands of *Callionymus* were fixed in Bouin's solution and embedded in paraffin; the sections (3 μ) were stained with haematoxylin-eosin.

For histological examinations of structure and activity of the thyroid glands LEVER's mathematical method (1948) was applied, which is based on the relations between the cell number, the outer and inner diameter and the height of the epithelium cells of circular or almost circular follicles. A short survey of this method, illustrated by figure 1, will follow here.

Fig. 1 A shows a graph in which the diameter of the whole follicle (outer diameter, upper line) and the diameter of the follicular cavity (inner diameter, bottom line) of an inactive thyroid of an untreated guppy is plotted against the number of epithelium cells.

In Fig. 1 B similar regression lines are constructed for a thyroid gland of a guppy, treated with thiourea. In both graphs the following details are visible:

1. In the inactive thyroid the vertical distance between the almost parallel regression lines is very small. This distance demonstrates the

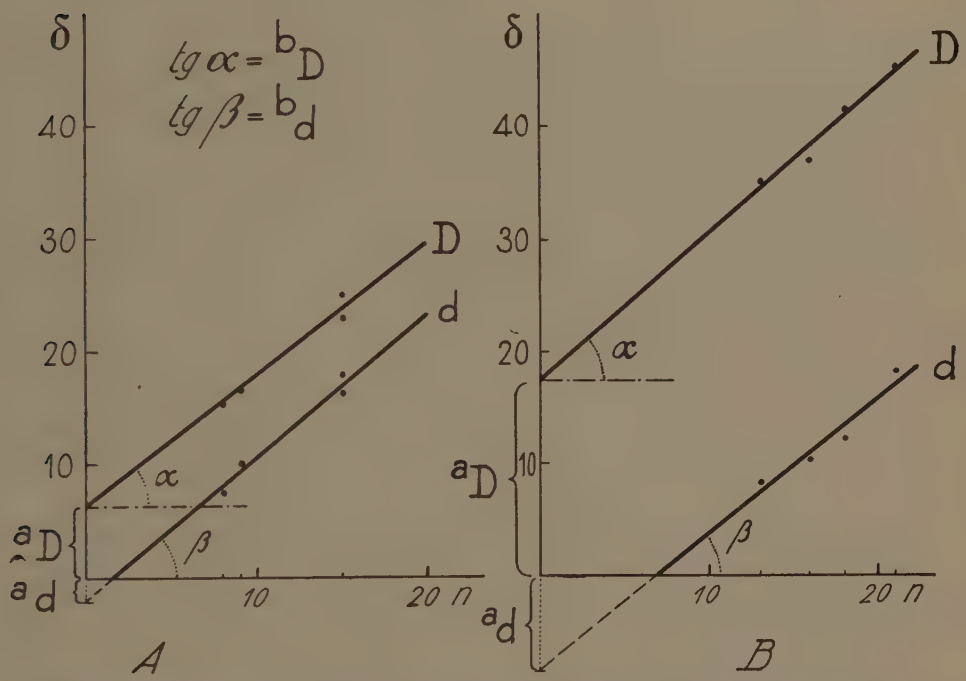

Fig. 1. Lebistes reticulatus. *A.* Graph, showing structure of thyroid follicles of an untreated control. *B.* The same of a specimen, treated with 0.1 % thiourea during 21 days. For coefficients cf. text.

difference between the outer and inner diameter and equals twice the height of the epithelium cells. In the active thyroid this distance is large and demonstrates the increase in height of the epithelium cells. By studying frequency diagrams of the outer diameter in hundreds of follicles of inactive and active thyroids of cockerels, treated with several antithyroid drugs during approximately 14 days, LEVER has found that the epithelium-height increases in both directions.

2. The inclination of the regression lines demonstrates the relation between the diameter of the follicles and the cell number.

3. The distance between the bottom line and the abscis demonstrates the colloid content in the follicles. As in the graph of the activated thyroid

the bottom line has moved to the right, the colloid content of the thyroid follicle decreases by activation.

The regression lines delineated in fig. 1 A and 1 B may be described by the formulae $D = a_D + b_D n$ and $d = a_d + b_d n$, in which D is the outer, d the inner diameter, n the number of cells of the follicle section, a_D and a_d the parts of the ordinate cut off by the regression lines and b_D and b_d the tangens of the angles of inclination. In inactive glands b_D and b_d are practically equal, as the regression lines run almost parallelly. Therefore in this case we can derive from these 2 lines one with the formula

$$Y = A + Bn \text{ in which } A = a_D - a_d \text{ and } B = \frac{b_D + b_d}{2},$$

A being related to the epithelium height and B to the number of cells dependent on the diameter of the follicle; if many cell divisions take place, the value of B decreases.

This method is easy to apply as sections stained by the ordinary laboratory routine techniques can be used; they must all have the same thickness; 3 μ-sections are the best, as these contain only one cell layer.

The results of our calculations are given in tables 1 and 2, from which the following conclusions may be drawn.

TABLE 1. Effect of thiourea on thyroid activity in Lebistes reticulatus (for coefficients cf. text).

	Duration of treatment in days	a_D	a_d	b_D	b_d	A	B
Untreated controls	7	8.7	4.1	0.8	0.8	4.6	0.8
	14	2.3	−6.1	1.5	1.4	8.4	1.5
	14	7.0	−2.7	1.4	1.4	9.7	1.4
	21	3.1	0.1	1.1	0.9		
	28	6.2	−2.0	1.2	1.3	8.4	1.3
	36	5.8	−0.7	1.2	1.1	6.5	1.2
0.01 %	8	9.8	−2.6	1.2	1.1	12.4	1.2
	14	11.1	−3.7	1.1	1.1	14.8	1.1
	14	4.5	−3.2	1.4	1.2		
	21	8.5	−3.9	1.4	1.1		
	28	5.1	−6.6	1.3	0.9		
	35	6.9	−0.3	1.3	1.3	7.2	1.3
0.1 %	7	4.6	−0.4	1.2	1.2	5.0	1.2
	12	2.7	−5.4	1.6	1.2		
	21	17.4	−8.4	1.3	1.2	25.8	1.3
	35	5.8	−1.5	1.9	1.0		
0.6 %	5	12.5	−0.8	1.0	1.0	13.3	1.0
	6	4.3	2.0	1.3	1.2	6.3	1.3
	21	8.1	−8.5	1.3	1.2	16.6	1.3

TABLE 2. Effect of thiourea on thyroid activity in Callionymus lyra (for coefficients cf. text).

	Duration of treatment in days	a_D	a_d	b_D	b_d	A	B
Untreated controls	0	7.9	— 3.0	1.6	1.5	10.9	1.5
	0	16.2	3.6	1.3	1.3	12.6	1.3
	6	—4.7	—10.3	1.5	1.4	5.6	1.5
	6	4.7	—10.2	1.2	1.2	14.9	1.2
0.01 %	1	16.9	0.5	1.2	1.1	16.4	1.2
	5	13.3	3.0	1.2	1.0		
	6	21.6	1.9	1.1	1.1	19.7	1.1
	7	12.5	2.1	1.2	1.1	10.4	1.2
	7	9.2	— 9.7	1.4	1.4	18.9	1.4
	7	16.5	— 3.8	1.2	1.0		
	9	13.4	—15.3	1.0	1.0	28.7	1.0
	11	4.4	—15.8	1.5	1.6	20.2	1.6
0.1 %	2	2.2	—13.9	1.7	1.6	16.1	1.7
	3	20.9	— 0.2	1.0	1.0	21.1	1.0
	7	12.3	— 6.8	1.3	1.3	19.1	1.3

1. A, i.e. the height of the follicle epithelium increases under the influence of thiourea in the concentrations used. This was especially distinct in *Callionymus*-specimens, treated with a 0.01 % solution of thiourea for only 7 days (fig. 2*a* and *b*).

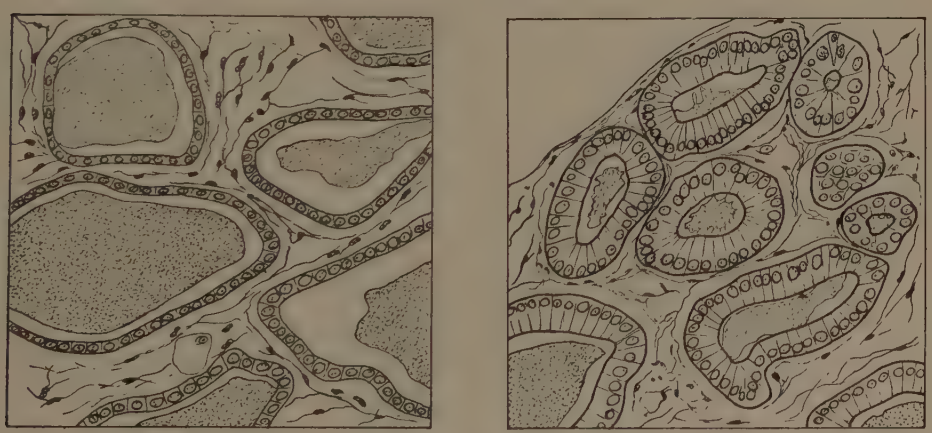

Fig. 2. Callionymus lyra. *a*. Section of a normal thyroid. *b*. Section of a thyroid of a specimen treated with 0.01 % thiourea during 7 days. (× 720)

2. B does not show a distinct change which points to the fact that cell divisions have not taken place.

3. a_D has the tendency to increase, a_d to decrease, which demonstrates

that also in these fishes the thickening of the epithelium has taken place in both directions, centrifugally as well as centripetally.

In addition it must be mentioned that the follicles generally decrease in size after activation, this being caused by colloid resorption and by the formation of new follicles by budding.

Summary.

From these data it may be concluded that as early as in the first week of treating *Lebistes reticulatus*, a freshwater fish, and *Callionymus lyra*, a marine fish, with thiourea-solutions of 0.01, 0.1, 0.6 and 0.01, 0.1 % respectively, an increase in the activity of the thyroid gland is observed.

Acknowledgement.

We wish to express our thanks to Prof. P. DRACH, Sous-directeur de la Station Biologique de Roscoff (Finistère, France) for providing us with *Callionymus*-specimens during our stay in his laboratory in September 1947.

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Zoology. — *An electron-microscopical study of bull sperm III.* By L. H. BRETSCHNEIDER. (From the Zoological Laboratory at Utrecht and the Netherlands Institute for Electron-microscopy at Delft.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of February 26, 1949.)

A. *Introduction.*

Following our first, more or less preliminary communication (1) on this subject, our investigation of bull spermatozoa was continued as one of the activities of the Unit for the study of Artificial Insemination, under the auspices of the National Council for Agricultural Research.

Investigations were again made with the aid of the electron microscope of the Institute for Electron Microscopy at Delft, the instrument having meanwhile undergone some improvements.

In a further communication (2), the measurement was described, by means of electron-microscopic photographs, of the different head-axes, and their ratios to one another expressed in indices, an "ideal" type being construed from the average values found.

The present, third communication purports to state various new and supplementary facts arrived at, partly by the use of electron-microscopic techniques specially adapted to this end. In addition to this we were able to clarify and rectify certain points in our first communication, which were based on incorrectly interpreted details. Owing to the, as yet, limited experience in the use of electron-microscopic methods, and the completely different manner of "reading" the images obtained by the electron microscope as compared to that with which we are familiar in the light-microscope, the investigator is often faced with a difficulty in coming to a decision, to be solved only by further research with the aid of different techniques.

B. *Techniques.*

As a full and detailed description of the techniques employed by us is yet to follow, we will confine ourselves here to enumerating the techniques used in treating the material dealt with in the present communication.

(1) To free the spermatozoa from the disturbing colloids of the spermoplasm, the ejaculated semen was rinsed 2—3 times with a phosphate buffer solution "400" according to ROMIJN (3). This solution contains, for 100 cc aq. dest., 0.4 g KH_2PO_4 (SÖRENSEN) and 1.954 g Na_2HPO_4. 12 aqu.

After this one of the following techniques was applied:

(2) Preservation of the spermatozoa in $\frac{1}{4}$—$\frac{1}{2}$ % mercurochrome (the disodium salt of 2.7 dibrom-4-hydroxymercurifluorescein) in aq. dest. during from 3 hours to several days.

(3) Treatment with 1 % $AuCl_4$ during 2 days.

(4) Treatment with 1/10—½ % chloramine-T (Sodium salt of toluene-sulfochloramide) in aq. dest., from 12 hours to 2 days, in darkness.

(5) Treatment with 0.005 % chromic acid in aq. dest., for 3 days at 40° C., in which a certain bacterial flora developed.

(6) Digestion with 0.3 % trypsin in 0.3 % Na_2CO_3, during from 2 to 48 hours at 38° C.; followed by rinsing in aq. dest. and fixing in OsO_4 2 %, 4 drops to 10 cc ½ % chromic acid for 24 hours.

(7) Bacterial maceration by developing the bacteria present in the semen, at room temperature, for 1—2 weeks, followed by rinsing in water and fixing in OsO_4-chromic acid as sub (6).

(8) Reagents for thymonucleic acid:

a) fixation in 2 parts concentrated corrosive sublimate (mercuric chloride) + 1 part absolute alcohol; 1 hour;

b) rinsing with alcoholic iodine solution, 1 hour;

c) lixiviation in 1 % "dimedon" in alcohol 90 % for 24 hours;

d) gradually reduced alcoholic sequence down to aq. dest.;

e) hydrolysis in 1 n HCl at 55° C. for 12 minutes;

f) thorough rinsing in aq. dest.;

g) impregnation in 10 % $AgNO_3$ solution, brought to a pH of 8—9 by a few drops of 8 % NH_4OH, after which the resultant precipitate is filtered away; 24 hours at 40° C.;

h) thorough rinsing in aq. dest. with a trace of NH_4OH to remove any remaining Ag_2O precipitate;

i) digestion in 1 % pepsin in 0.1 n HCl, at 40° C. for 24 hours.

Following all these reactions, and before commencing the electron-microscopic examination, repeated rinsing in aq. dest., in which the objects then come on the container of the preparation.

Some preparations were shadowcast with gold according to WILLIAMS and WYCKOFF (4). The electron accelerating voltage varied between 80, 90 and 100 kV.

C. *Examination of the sperm head.*

(*a*) *The head-cap.* Given efficient fixation, it is possible to demonstrate the fragile head-cap electron-microscopically.

In the living state the substance of which this cap consists is rich in water, so that, after the necessary dehydration, next to nothing remains of it. As will be seen from fig. 1 and 2 in our first communication (1) nothing much can be seen of it after treatment with proteolytic agents. For this reason we had to find a fixative, which we found in mercurochrome (vide above, technique 2) (2). The sublimate component of the mercurochrome causes the substance of the cap to coagulate, so that it is preserved in its original shape, also in a vacuum (fig. 1, 2). The thickness of this cap, i.e. the distance between the outer edge and the head-profile, measures, in these preparations, from 250—370 mμ. The structure of the

outer part is slightly denser than the inner zone, which usually contains vacuoles and small granules. The vacuoles measure ± 80 mμ, the granules ± 50 mμ.

(*b*) *The distribution of the head-plasma.* The occurrence of certain zones and vacuoles in the sperm-head is probably a result of a corresponding distribution of a so-called head-plasma. The head of a normal bull sperm does not show any vacuoles within its membrane. Whenever such vacuoles are visible, we must certainly have to do with artifacts. Vacuoles may occur through autolysis of dead spermatozoa in the ejaculated semen, or by the treatment with substances which dissolve the content of the head; such vacuoles probably originate from the products of disintegration of the contents of the head. Especially after proteolysis by bacterial enzymes (fig. 3, technique 7), or after chloramine (fig. 1 in our first communication), numerous vacuoles appear. These vacuoles measure from 300—500 mμ, are generally sharply outlined, and may break through the membrance of the head when the latter is treated with a proteolytic. Owing to lack of adequate material for comparison, we took such a burst-open vacuole to be a "porus" (Communication 1, p. 94), which erroneous interpretation we hereby rectify.

What also strikes one in fig. 3 is the "preference" of these vacuoles for a certain position in the head. They accumulate especially below the equator, in a straight line, and also at the apex of the head. We examined statistically the number and position of the vacuoles in 50 other heads in which vacuoles were found, and found the following distribution curve (fig. *A*). If one divides the head into 7 zones, then it will be seen that

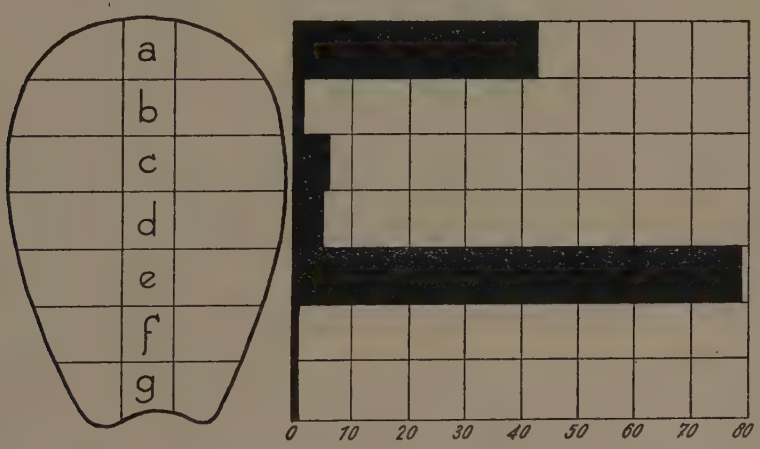

Fig. A. Frequency of the occurrence of vacuoles in the head.

most of the vacuoles occur in zone *a* (at the apex) and zone *e* (below the equator), while they only occur sporadically in the remaining zones. In our first communication (1) we already pointed to different zones, especially the lighter one in the equator which, in this case, appeared after the treat-

ment with chloramine-T. We now see that a similar zonal distribution also takes place after the action of proteolytic enzymes, from which one might conclude that the products of the disintegration collect on the spot and there form vacuoles. In correspondence with this assumption we see that in this equatorial zone, which has a width of 400—600 mμ, the chromatinic material is practically lacking (fig. 4). We may conclude from these facts that there is an apical and, especially, an equatorial accumulation of nuclear plasma whose lysis or dissolution gives rise to vacuoles.

(c) *The chromosomes.* Apparently, the chromosomes or their chromatinic derivatives can be demonstrated electron-microscopically in the head of the bull sperm by means of impregnation with silver.

When the staining method to thymonucleic acid according to FEULGEN is applied light-microscopically, the whole of the head is coloured an even red with parafuchsin, only the base being slightly darker. It was concluded from this that the chromatin filled the whole of the head as a compact mass. It was to be expected that the so much greater magnification possible with the electron-microscopic method would enable us to get to know more about the distribution of the chromatin. To this end, however, dyes are required with a greater electronic dispersion than that of the organic ones. For this reason we applied a variant of FEULGEN's reaction, i.e. replacing the parafuchsin by an ammoniacal silver nitrate solution (technique 8). In this, the Ag_2O is reduced by the lytic products of the thymonucleic acid, the Ag being precipitated in the nucleus. Fig. 4 shows the result of the electron-microscopic examination of sperm heads treated by technique 8. We then find the places, where the thymonucleic acid was, to be spread all over the head, in the form of smaller or larger granules. The densest concentration occurs in the base of the head; the smallest in the almost granule-free zone of equator, which corresponds to the nuclear plasma zone already mentioned above.

Although in the majority of cases the chromatin is seen to be very widely dispersed, we still find groups of granules, particularly in the front half of the head, which lie together in shorter or longer rows. One has the impression that parts of chromosomes are still present here. Closer analysis generally shows that in these parts pairs of granules — which might be chromomeres and chromioles — are linked up into a chain; they resemble parts which have not dispersed as widely as the rest. Since the size of these granules varies between 35 mμ and 260 mμ and their spatial distribution is very dense, it is understandable that they give only an evenly-coloured picture at the enlargements of the light-microscope. Not in all spermatozoa is the chromatin so finely distributed; sometimes the chromomeres still lie together in chromosome-formation. In view of the large number of chromosomes — according to KRALLINGER (6) 30 in the haploid form, — the chromosomes individually can not be so very long. Comparison of the configuration of the chromatin in the sperm head with that of the nucleus in the metasperm stage (vide 17) reveals a considerable similarity:

in all cases the chromatin formations lie together very compactly in the base of the head; a few series of granules pass the equatorial plasma zone, after which a looser spread of the granules or parts of chromosomes is seen in the front half of the head.

D. *Examination of the sperm tail.*

(*a*) JENSEN's *"spiral body"*. By this is meant the spiral ("broad helix", see our first communication) in the middle piece, originating from the mitochondria. Owing to its being rich in lipoids it can be convincingly demonstrated electron-microscopically by appropriate techniques.

JENSEN's spiral body consists of two equally thick spirals running parallel with a pitch of 15—25 degrees around the fibrillary axis. In our first communication we were able to identify the spiral body only fragmentarily and after chloramin treatment and shadow-casting. Starting from the assumption that this structure might in some way be connected with the generation of energy, we endeavoured to demonstrate it by different electron-microscopic methods. When the middle piece is examined without preliminary treatment, the spiral windings are observable only as slightly darker strips (fig. 5). Application of technique No. 5 causes the spiral body to be more distinctly outlined, bacterial enzymes, coupled with the chromatization of the lipoids rendering the spiral-like structure plainly visible. Here too it is seen for the first time that the structure consists of two spirals, as, owing to the greater pitch of 25 degrees the distance of the windings increases in different places. The real thickness of each spiral band, too, is more easily measured here, i.e. about 170 mμ.

(*b*) JENSEN's *ring*. The electron-microscopic image shows a ring-formed structure at the transition between the middle piece and the tail. This structure, called after its discoverer CARL OLUF JENSEN (1887), is impregnated with Au ions (fig. 6). The diameter of the ring, which is 150 mμ thick, is usually slightly wider than the diameter of the middle piece, with the result that the tail here begins with its own, broader base, and no direct transition exists from the middle piece to the tail.

A number of pathologically changed sperms, which are described in more detail further on, confirm the independent position of this caudal structure.

(*c*) *The cortical helix (tail-spirals)*. These can be demonstrated electron-microscopically and without shadow-casting, after impregnation with OsO_4.

After treatment with bacterial enzymes, whose action is less radical than that of either trypsin or pepsin, the tail plasma is dissolved, after which the remaining axial fibrils and the cortical spirals become visible by impregnation with OsO_4 and chromic acid, which gives a sharper contrast (fig. 7).

Already in 1947 we gave it as our opinion, from the manner in which these spiral fibrils, also in other images, generally shift from their original position in groups of 3 — vide the accompanying sketch (fig 7), that there

are three spiral fibrils running around the tail. As the coarsening action of the gold particles after shadow-casting is eliminated here (vide fig. 10 of 1), a more exact measurement of the thickness of the fibrils is possible. Each fibril is about 30 mμ thick, and makes about 150 spiral windings around the tail.

(*d*) *The terminal piece* consists of the 9 sub-fibrils which, bound together to form a rod-like axis, protrude from the body of the tail. Usually this piece is slightly curved or sharply bent at the end (fig. 9). Either the quick dehydration or proteolysis causes this terminal piece to fall apart to form an artifact resembling a tiny brush (fig. 8). Whereas the intact terminal piece has equally long sub-fibrils, the latter, after unravelling, break off artificially at different lengths. The subfibrils can be demonstrated also without shadow-casting and preliminary treatment, as fig. 8 shows. No definite structure can be observed to finish off the tail spirals; the spiral fibrils here finish abruptly.

E. *Discussion and Summary.*

However praiseworthy may be the endeavours made, in applying electron-microscopy, to examine the objects without any preliminary treatment after necessary dehydration only, it appears that this ideal is difficult of realization in the case of biological objects.

The desirability of the development of a special electron-microscopic preparation technique has been understood from the very beginning at the Netherlands Institute for Electron-Microscopy. With many objects, the inner structure is obscured by their thickness, as they give uniformly black images, unless measures are taken to dissolve and remove part of their substance. When the preparation transmits the electron rays, a fruitful analysis is often prevented by the lack of contrasting atoms having a higher atom number than those which compose the organic substance.

The structures described above, such as the head-cap, the chromosomes, the double spiral body and JENSEN's ring were rendered visible only after certain technical expedients had been applied. This indeed makes the further development and elaboration of a special electron-microscopic preparatory technique desirable. The electron-microscopic investigation of the spermatozoon forms the immediate sequel to light-microscopic examination at the point where the latter has long since reached its extreme limit of applicability, culminating in the work of RETZIUS (7). Already in light-optical observation in the range of 1/10 micron, subjective guessing often takes the place of objective observation.

Electron-microscopic examination not only eliminates the necessity for such guessing but also reveals structural details that would be invisible light-microscopically. As such we may mention, in the present investigation, the following: the granules and vacuoles in the head-cap, of 50—80 mμ; the distance between the two spiral bodies, of 10—30 mμ; the "chromioles", 35—100 mμ, and the cortical fibrils, 30 mμ. In this way the continued

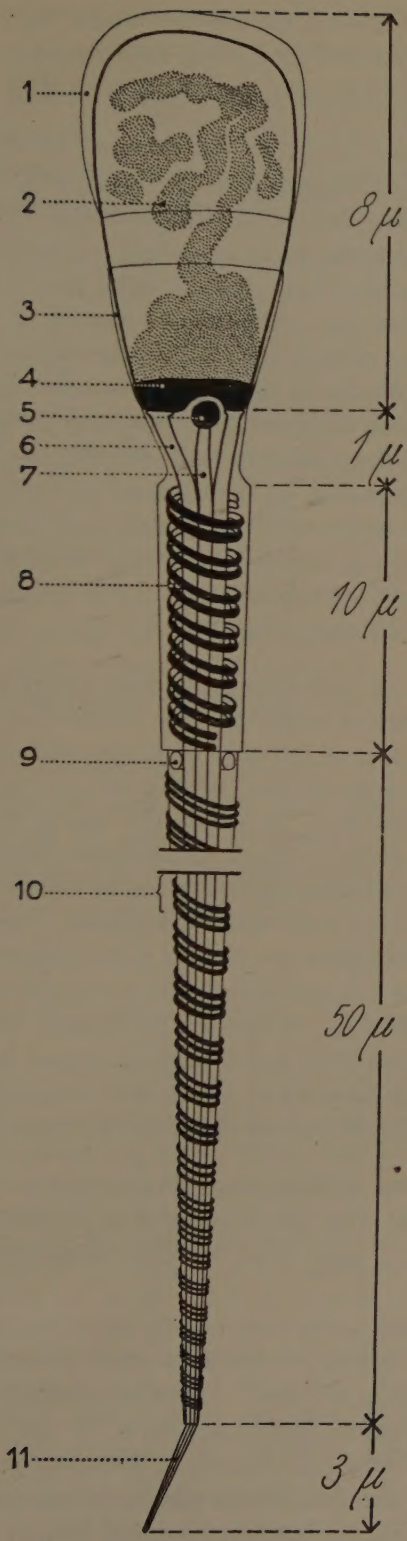

Fig. B. Diagram of the structure of the sperm.

1. head cap;
2. chromosomes;
3. head tunica, (external layer);
4. ring-shaped membrane;
5. centrosome;
6. articular strands;
7. axial filament;
8. double broad helix (JENSEN's spiral body);
9. Jensen's ring;
10. cortical helix;
11. terminal piece.

development of the investigation and the new details discovered causes us to modify our conception of the structure of the sperm, enabling us to add the following supplementary structures to our original scheme (fig. *B*).

1. *Chromatin,* partly still to some extent in the form of individual chromosomes, or dispersed in the form of smaller or larger fragments, down to chromiole formation. It is interesting to note in this connexion that on account of the duplicity of the chromatin parts many of the fragments appear to be composed of 2 chromatids. From the similarity in the configuration of the chromatin in the spermatid nuclei and the sperms we may conclude that chromosomes, or parts thereof persist in this interphase. That chromosomes are retained in the sperm-head is known also of the lower animals. Thus, we see in the case of Nematoda such as *Ancyracanthus,* according to MULSOW (8), the chromosomes already *in vivo.* SOKOLOW (9) found, in different arachnids, by light-optical methods, the chromosomes in various degrees of condensation or dispersal. POLLISTER and MIRSKY (10) distinctly observed the presence of chromosomes in the dilating sperm-head of the trout, after treatment with 1/M NaCl. Not until now has this been possible in the case of mammals.

2. *The head-plasma,* particularly as a separate zone in the centre of the head. It is known that the sperm contains relatively little water. The head is considerably reduced in size in respect of the original spermatid nucleus; and it seems feasible that nothing more remains of the original nuclear fluid, after this condensation of substances composing the sperm, than a more or less condensed gelatinous colloid — the so-called head plasma, in which the chromatin is embedded. In addition the substance of the nucleoli, after their disintegration is incorporated with this head-plasma. Through the dissolution of the head-plasma, vacuoles are formed which burst open outwardly. In our first communication (1) we erroneously interpreted these artifacts as "pores". WILSON (11) already put forward the theory that the light-optically visible homogeneous and solid nuclei of most sperms largely consist of substances from the nuclear framework which, after insemination cause the nucleus to swell again owing to the incorporation of water.

3. *The double spiral body and the middle piece.* The extent to which the principle of the double spiral body is carried through in the spermatozoa of the vertebrates can be ascertained only after the electron-microscopic examination of other spermatozoa.

Light-optically only a single spiral body was known to us up to the present — from RETZIUS' investigations (7); but it is possible nevertheless that the distance between both spirals is too small for the resolving capacity of the light-microscope. Of invertebrates, double spirals in the tail of the sperm are already known; thus, WILSON (12) found, in the scorpion *Centrurus,* two strands originating from the chondriome which, during spermiocytogenesis transform themselves into two intertwined spiral bodies.

BÖHMING (13) found a similar double spiral in the Turbellaria *Plagiostoma sulphureum* and *Plag. maculatum*.

4. JENSEN's *ring*. By its stronger colour it drew JENSEN's attention as early as 1887 (14), since when it has been regarded as a derivative of the hindmost centrosome. It is a disc pierced by the fibrillary axis, and forms the border between the middle piece and the tail.

5. *The normal terminal piece*. Our more detailed and extensive examination proved that the brush-like terminal piece described, in the case of the bull, by BAYLOR, NALBANDOV and CLARK (15), and ourselves (1), is in effect based on an artifact, since in the majority of sperms, or after preliminary fixation, it is found to consist of a fixed axis composed of 9 subfibrils. In the majority of sperms we have so far examined, the terminal piece was also shown to consist of a solid bundle of fibrils.

A renewed examination of the cortical tail spirals finally confirmed the interpretation already put forward in 1947, to the effect that there are, in fact, three fibrils present which run side by side. Such a threefold spiral was also found by KOLZOFF (16) in the spermatozoa of *Planorbis* by causing the sperms to dilate; our own findings in the case of the bull, therefore, constitute a sequel to KOLZOFF's discovery and point to the existence of a more widely spread principle.

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